STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2015-16 & thereafter)

SUBJECT CODE : 15MT/PC/MI24

M. Sc. DEGREE EXAMINATION, APRIL 2018
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : MEASURE THEORY AND INTEGRATION
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

Answer all the questions: **5×2=10**

1. Show that every countable set has measure zero.
2. Given an example of a function such that $|f|$ is measureable but f is not.
3. When do you say a function f defined on $(-\infty,\infty)$ is Riemann integrable.
4. Write any two examples of functions that are strictly convex on R.
5. Define mutually singular measures.

SECTION – B

Answer any five questions: **5×6=30**

6. Prove that every interval is measurable.
7. Let T be a measurable set of positive measure and let $T^* = [x-y: x \in T, y \in T]$. Show that T^* contains an interval $(-\alpha,\alpha)$ for some $\alpha > 0$.
8. Let f and g be non-negative measurable functions. Prove that $\int f dx + \int g dx = \int (f+g) dx$.
9. Let ν be a signed measure and let μ, λ be measures on $[[X,S]]$, such that λ, μ, ν are σ −finite, $\nu \le \mu$ and $\mu \le \lambda$, then prove that $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} . \frac{d\mu}{d\lambda} [\lambda]$.
10. Prove that every function convex on an open interval is continuous.
11. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \le g$, where g is integrable and let $\lim f_n = f \ a.e.$ Prove that f is integrable and $lim \int f_n dx = \int f dx$.
12. Prove that a countable union of sets positive with respect to a signed measure ν is a positive set.

SECTION – C

Answer any three questions: **3×20=60**

13. (a) Prove that the outer measure of an interval equals its length.

(b) Prove that if $m^*(E) < \infty$ then E is measurable if, and only if, for every $\varepsilon > 0$, there exists disjoint finite intervals $I_1, \ldots, I_n$ such that $m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \varepsilon$.

14. (a) Let c be any real number and let f and g be real valued measurable functions defined on the same measurable set E. Prove that $f + c$, $c\,f$, $f + g$, $f - g$ and $f\,g$ are also measurable.

(b) If μ is a measure on a $\sigma - ring\ S$, then prove that the class $\bar{S}$ of sets of the form $E\Delta N$ for any sets E, N such that $E \in S$ while N is contained in someset in S of zero measure, is a $\sigma - ring$, and the set function $\bar{\mu}$ defined by $\bar{\mu}(E\Delta N) = \mu(E)$ is a complete measure on $\bar{S}$.

15. (a) State and prove Fatou's lemma.

(b) Let f be a bounded function defined on the finite interval $[a, b]$. Prove that f is Riemann integrable over $[a, b]$ if, and only if, it is continuous $a.e.$

16. (a) If $1 \leq p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $||f_n - f_m||_p \to 0$, $as\ n, m \to \infty$, then prove that there exists a function f and a subsequence $\{n_i\}$ such that $lim f_{n_i} = f\ a.e$. Also prove that $f \in L^p(\mu)$ and $\lim ||f_n - f_m||_p = 0$.

(b) If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure then prove that there exists a measurable function f such that $f_n \to f$ is measure.

17. (a) Let ν be a signed measure on $[[X, S]]$. Let $E \in S$ and $\nu(E) > 0$. Prove that there exists A, a set positive with respect to ν, such that $A \subseteq E$ and $\nu(A) > 0$.

(b) State and prove Lebesgue Decomposition theorem.