INTRODUCTION

The study of superharmonic functions is closely related to the study of analytic functions in the complex plane, mainly because, for an analytic function f in $\mathbb{C}$, $\log\{1/|f|\}$ is superharmonic. Our purpose in this thesis is to discuss the superharmonic analogues of some results on analytic functions.

The extensions from analytic functions to superharmonic functions are considered in three different stages; from analytic functions in $\mathbb{C}$ to superharmonic functions in $\mathbb{R}^2$, then from superharmonic functions in $\mathbb{R}^2$ to superharmonic functions in $\mathbb{R}^n$ $(n \geq 3)$ and finally from $\mathbb{R}^n$ and Riemann surfaces to the axiomatic harmonic spaces. The need to deal with the cases corresponding to $\mathbb{R}^2$ and $\mathbb{R}^n$ $(n \geq 3)$ separately, arises owing to the fundamental difference in the harmonic structures of $\mathbb{R}^2$ and $\mathbb{R}^n$ $(n \geq 3)$. The analogues in the axiomatic case, which are of more general nature, are obtained after careful considerations, since many of the simple results in $\mathbb{R}^n$ are not easily available in harmonic spaces.

Some interesting notions connected with analytic functions, the analogues of which we consider, in our studies with respect to superharmonic functions are : order, number of zeros, canonical decomposition, Green functions and analytic maps from a Riemann surface into another.

CHAPTER I : MINIMAL THEOREMS FOR THE HARMONIC-MORPHISM

The theory of 'analytic maps', we know, plays a key role in determining the geometric properties of analytic functions in the complex plane. These analytic maps have also been studied in the context of Riemann surfaces in detail. In fact, the study of analytic maps have been extended (C.Constantinescu and A.Cornea [11]) to more comprehensive structures which are defined by means of a sheaf of continuous functions on topological spaces, possessing the main properties of the sheaf of harmonic functions on a Riemann surface. This leads to the notion of harmonic-morphisms (also referred to as harmonic maps) between harmonic spaces.

In this chapter, we study the action of harmonic-morphisms on certain types of positive harmonic functions. The underlying space of our study is a B.P. harmonic space, denoted by Ω.

Imitating the developments in the theory of Riemann surfaces, (M.Heins [22]) we first characterise Bl-harmonic-morphisms using singular harmonic functions. In these characterisations, we make use of the 'upper limit functions' associated with subharmonic functions, which we introduce, also deriving a few of their properties.

The key theorems of the chapter can be classified under what we call as 'minimal theorems'. Such theorems in the classical case have been discussed by M.Heins [25]. Let $\phi : \Omega_1 \to \Omega_2$ be a harmonic-morphism and u be a real valued function in Ω_1. We denote by F^u, a class of real valued functions s in Ω_2 satisfying $u \leq s \circ \phi$. The minimal theorems consist in investigating the nature of the infimum of the family F^u, under restrictions placed on u, F^u and the map ϕ.

CHAPTER II : TWO APPLICATIONS OF THE ADJOINT HARMONIC SHEAF

The properties of the Green functions in a bounded domain in $\mathbb{R}^n$ and in Green spaces (M.Brelot [10]) are extensively used to obtain many of the well known results involving harmonic and superharmonic functions in such spaces. Is it possible to lift these functions to the axiomatic case? In fact the potentials in B.P. spaces with point support, mostly come up to our expectations, but with symmetry lacking in them. To compromise for the non-symmetry of these functions, we go over to the adjoint harmonic sheaf.

In this chapter, we illustrate, by considering two problems, how potentials with point support in B.P. spaces and in the adjoint sheaves successfully lead to generalisations of some known results in the classical theory.

The first problem we consider, is a characterisation of regular points for the Dirichlet problem in B.P. spaces. The Dirichlet problem, which is one of the earliest boundary value problems in Analysis, has played an important role in the whole development of Potential Theory. Its classical formulation is well known. 'Given a continuous function on the boundary of an open set, to find a harmonic function tending to the prescribed boundary condition at every point . Since the problem posed in the above manner was not always solvable, it was thought in the earlier part of the century that the study of the Dirichlet problem should be divided into two steps. First to assign in a good way a harmonic function h_f to the boundary data f and then to investigate the boundary behaviour of h_f and compare the result with values of f. This led to the notion of the generalised Dirichlet problem and points of the boundary were classified as regular and irregular, depending upon the well behaviour of the Dirichlet solution at these points.

There have been various characterisations given for the regular points of the Dirichlet problem, for example in terms of barrier functions, thinness of the complement, (M.Brelot [9]) and recently using Keldysch operators. (Netuka, [34]). In our discussions, we intend giving a characterisation for regularity in terms of the quasi-boundedness of the potential with point support in the adjoint harmonic sheaf. In the special case of self-adjoint harmonic spaces, this essentially reduces to quasi-boundedness of the potentials with point support in the space under consideration.

The second problem we consider, deals with the existence of harmonic majorants for subharmonic functions. It is known that (Gauthier and Hergartner [13]) if u is subharmonic in the open unit disc in $\mathbb{R}^2$ and if each point of the unit circle has a neighbourhood on which u has a harmonic majorant, then u has a (global) harmonic majorant in the disc. It is essential in such a theorem, to consider boundary points, for, every subharmonic function has a local harmonic majorant in the neighbourhood of every interior point, and of course, many subharmonic functions do not have global harmonic majorants.

In the axiomatic case, the Martin compactification $\widehat{\Omega}$ of a B.P. space Ω, called the 'Martin space', behaves like the unit disc in $\mathbb{R}^2$, in considerations regarding harmonic functions. It is but natural therefore, to anticipate a relationship between the existence of local harmonic majorants in neighbourhoods of Martin boundary points and the existence of global harmonic majorants, for subharmonic functions in B.P. spaces. We prove that in a Martin space which is also locally connected at the boundary and with every boundary point minimal, every subharmonic function with the local harmonic majorant property in Ω, has a global harmonic majorant. Recalling that a harmonic function u belongs to the Hardy class h^p, $(0 < p < \infty)$, (L.Lumer, [32]) if $|u|^p$ admits a harmonic majorant, we also make the following interesting deduction that in a Martin space with the above mentioned additional conditions, a harmonic function is globally in a Hardy class if and only if it is locally in the same Hardy class.

CHAPTER III : SUPERHARMONIC EXTENSIONS AND FLUX AT INFINITY

In this chapter, our underlying space of study is a B.S. harmonic space Ω .

The measure associated with a superharmonic function, is the counterpart of the number of zeros of an analytic function in $\mathbb{C}$. Special interest arises, with superharmonic functions, whose associated total measure is finite. In B.S. harmonic spaces, these functions correspond to the admissible superharmonic functions, which have their flux at infinity, finite.

'Flux at infinity', is a notion that generalises the concept of finite total measure associated with a superharmonic function, in the classical case, and is closely related to questions concerning the extensions of superharmonic functions.

In the theory of superharmonic extensions, we often encounter the following basic theorem, that leads to the definition of flux at infinity. " Given a superharmonic function u in a domain, in a B.S. space Ω, with compact support K, and a harmonic function h outside a compact set, there exists a function s in Ω, harmonic outside K, such that $s = \lambda u +$ a harmonic function in a neighbourhood of K, for some constant λ and s-h is bounded outside a compact set in Ω" .

The proof of the above theorem, (V.Anandam [1]) has, as its key note, the existence of a solution of the equation $(1-T)g = f$, where f and $g \in C(\partial X)$ for a suitably chosen relatively compact set X in Ω and T is a normal operator on $C(\partial X)$. (See Preliminaries A). In B.P. spaces it is known that (Rodin and Sario [43]) the above equation always has a solution for every $f \in C(\partial X)$. But this is not the case with B.S. spaces. In these spaces the above equation has a solution if and only if $\int f \, d\nu = 0$, where ν is a unique Radon measure on $C(\partial X)$ determined by $\int \phi \, d\nu = \int T\phi \, d\nu$, for every $\phi \in C(\partial X)$. The above result on the existence of the Radon measure is known as 'Nakai's lemma', the proof of which entirely depends upon the Riesz-Schauder theory of operators. One understands from the above cited results, that the extension problems in B.S. spaces are not so easy to handle with, as in the case of B.P. spaces.

The discussions initiated by V.Anandam [1] using Nakai's results are thus essentially based upon the Riesz-Schauder theory of operators. Attempts have been made to provide solutions to the above theorem and allied ones, without applying Riesz-Schauder theory. In fact, Anandam [3] has given direct proofs for theorems on superharmonic extensions, in harmonic spaces of dimension one. Guillerme, [19] has proved an extension theorem for harmonic functions defined outside a compact set, without making use of Riesz-Schauder theory. Guessous, [17] has considered both the cases of extensions from inside and outside without any operator theory.

In this chapter, we prove the above mentioned theorem using only potential theoretic techniques, thus eliminating Riesz-Schauder theory completely, and deduce a few other related results. We mainly use the concept of 'specific majorisation' and the properties of 'Specific reduced functions' in the course of our proofs. Also, we indicate how our results lead to the definition of flux at infinity.

CHAPTER IV : GENERALISED CLASSICAL KERNELS IN HALF-SPACES :

As indicated earlier, this chapter is devoted to discussions in the classical case.

It is known that given a Radon measure $\mu > 0$ in $\mathbb{R}^n$, there exists a superharmonic function u whose associated measure in the local representation is μ. Though in general, nothing more precise, can be said about u thus obtained, when μ is restricted in some sense, then u assumes nice properties. The restrictions on the measure lead to the notion of genus and correspondingly, superharmonic functions with finite genus, admit integral representations in terms of canonical potentials and harmonic functions. The kernels involved in the canonical potentials are referred to as the generalised kernels. (See V.Anandam and M.Brelot [4], W.K.Hayman [21] and Premalatha and V.Anandam [37]).

How do the above results modify in half-spaces D^n in $\mathbb{R}^n$? It turns out that one has to consider two sets of kernels, called generalised superharmonic kernels and generalised Poisson kernels, in the integral representation of superharmonic functions in D^n. These kernels have been considered by D.H.Armitage [5], [6], in the context of integral representations.

In this chapter, we supplement Armitage's results, giving conditions for the existence of generalised potentials, and generalised harmonic functions, in terms of the given superharmonic function and its associated measure. Imitating, the developments in the study of superharmonic functions in $\mathbb{R}^n$, we define the order of a superharmonic function, the genus and convergence exponent of the associated measure μ in the half-space D^n and obtain inequalities connecting them. Also, we interpret the integral representation theorem of Armitage, as the superharmonic version of the 'Hadamard theorem' involving the order and genus. We end the chapter by briefly indicating how some of the integral representations in half-spaces can be carried over to wedges in $\mathbb{R}^n$ and in particular, wedges in $\mathbb{R}^2$.

CHAPTER V : GROWTH OF SUPERHARMONIC FUNCTIONS IN $\mathbb{R}^2$

The order of a superharmonic function u and the exponent of convergence of its associated measure μ are mainly introduced with a view to study the behaviour of the function and the measure near the point at infinity. However the growth of u near the point at infinity is not fully independent of the growth of μ. We assert this, in this small chapter, for superharmonic functions of non-integral order, by establishing that u is of regular growth if and only if μ is of regular growth. As it usually happens, superharmonic functions of integral order pose some difficulties, in obtaining a result of the above type for them. However, we are able to guess that at least in the case $g = \lambda - 1$, the regular growth for u need not be a consequence of the regular growth for μ.

PART A

DISCUSSION IN AXIOMATIC HARMONIC SPACES

P R E L I M I N A R I E S - A

Here, we recall briefly, the fundamental axioms of the axiomatic theory and certain results we use in our discussions. The details are to be found in M.Brelot [9], Mme.Herve [26] and V.Anandam [1].

Let Ω be a locally compact, non-compact, connected and locally connected Hausdorff space. To each open set in Ω, is assigned a vector space of finite continuous real valued functions called, ' harmonic functions' on this set.

An open set $V \subset \Omega$ is called regular, if it is non-empty, relatively compact and for every finite continuous function f on the boundary ∂V of V, there exists a unique harmonic function H_f^V on V such that H_f^V tends to f at each point of the boundary and $f \geq 0$ implies that $H_f^V \geq 0$. For any $x \in V$, the functional $f \longrightarrow H_f^V(X)$ is a non-negative Radon measure ρ_X^V on ∂V, called the harmonic measure relative to V and x.

We assume that the system of harmonic functions on Ω satisfies the following 3 axioms of M.Brelot.

Axiom 1 : The harmonic functions have the sheaf property.

Axiom 2 : Ω has a base consisting of connected regular open sets.

Axiom 3 : The upper envelope of an increasing family of harmonic functions is either identically infinite or is harmonic.

We also assume that the constants are harmonic in Ω. We call Ω