C H A P T E R I

INTRODUCTION AND PRELIMINARIES

A region in the complex plane denoted by $\mathbb{C}$ is an open connected set. A function f regular in a region D is called p-valent if no value of the function is taken more than p times in D and if atleast one value of the function is taken exactly p times. When p is unity the function is said to be univalent or schlicht in D. In this case, f never takes in D the same value more than once, that is, $f(z_1) \neq f(z_2)$, $z_1 \neq z_2$, $z_1, z_2 \in D$. By the celebrated Riemann mapping theorem which states that there exists a holomorphic univalent function mapping an arbitrary simply connected region in $\mathbb{C}$ other than the whole plane (that is, having atleast one boundary point) onto the unit disc, given any simply connected region D in $\mathbb{C}$ and a univalent analytic function defined on D we can always associate with it, one defined in the unit disc. Thus the unit disc can be taken as the domain of investigation of univalent functions. Further, if f is univalent in D then so is the function F defined by $F(z) = (f(z) - f(0)) / f'(0)$, since the derivative of a univalent function does not vanish in D. This observation is used to normalise the function f by the condition $f(0)=0; f'(0)=1$

and as such, any property of f can be immediately translated to the corresponding property of F. Therefore without loss of generality the study of univalent functions can be confined to the study of the normalised class of univalent functions regular in the unit disc. The introduction of the normalised class of univalent functions facilitates computations and leads to simple, elegant results. Moreover, the normalised class is compact in the space of regular functions defined in the unit disc, endowed with the topology of uniform convergence in compact subsets.

The following notations are used throughout the thesis. $\mathbb{C}$ denotes the complex plane, $E = \{z : |z| < 1,\ z \in \mathbb{C}\}$ the open unit disc, H the class of regular functions f(z) in E normalised by the conditions $f(0) = 0 = f'(0)-1$ and S the subclass of H consisting of univalent functions. Let P denote the class of regular functions p(z) in E with $p(0) = 1$ which map E into the right half plane characterised by the condition $\operatorname{Re} p(z) > 0$, $z \in E$. This class plays a very important role in the study of univalent functions. The various geometrical properties which arise in the study of special regions like convex, starlike, close-to-convex, spiral-like and regions having bounded boundary rotation can be completely characterised by equivalent analytic formulations with the help of the members of the class P. This class is equipped with interesting representations like Herglotz formula and representation in terms of unit functions (regular functions bounded by unity in E) which are given below. First we introduce the notion of subordination and state a well known lemma.

Definition (1.1.1). Subordination [34]. If f and g are analytic in E and $g(z) = f(w(z))$, where $w(z)$ is analytic and satisfies $|w(z)| \leq |z|$ in E, then g is said to be subordinate to f in E. In symbols we express this by $g \prec f$. When f is univalent in E, this is analytically equivalent to the statement $g(E) \subset f(E)$, $g(0) = f(0)$.

Theorem (1.1.2). Schwarz's Lemma [34]. Let f be a holomorphic function in the unit disc and suppose that $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. Then

(i) $|f(z)| \leq |z|$ for $|z| < 1$,

(ii) if for $z_0 \neq 0$ the equality $|f(z_0)| = |z_0|$ holds then $f(z) = Az$ where $|A| = 1$.

Theorem (1.1.3). Herglotz formula [44]. The function p belongs to P if and only if there exists an increasing function $\gamma(t)$ $(-\pi \leq t \leq \pi)$ such that

$$p(z) = \int_{-\pi}^{\pi} \frac{1+e^{-it}z}{1-e^{-it}z}\, d\,\gamma(t), \quad \gamma(\pi) - \gamma(-\pi) = 1.$$

<u>Theorem (1.1.4)</u> [34]. The function p belongs to P if and only if there exists a function w(z) analytic in E satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$, such that

$$p(z) = \frac{1+w(z)}{1-w(z)} .$$

Since the function $w = (1+z)/(1-z)$ which is a member of P maps the disc E onto the right half plane $\mathrm{Re}\, w > 0$, it follows immediately from the definition of the class P and the concept of subordination that a function p belongs to the class P, if and only if $p(z) \prec (1+z)/(1-z)$ on E. This representation for members of P in terms of unit functions motivated Janowski [20] to introduce a new class of functions which we denote by P(A,B).

<u>Definition (1.1.5)</u> [20]. A function p(z) analytic in E belongs to the class P(A,B), $-1 \leq B < A \leq 1$, if and only if, $p(z) = (1+Aw(z))/(1+Bw(z))$, $z \in E$ where w(z) is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ in E.

Clearly, the class P(+1,-1) coincides with the class P and a function p(z) belongs to P(A,B) if and only if, $p(z) \prec (1+Az)/(1+Bz)$ in E. Further, since the transformation $T(z) = (1+Az)/(1+Bz)$ maps the region $|z| \leq r$ univalently onto the disc with centre at $(1-ABr^2)/(1-B^2r^2)$ and radius $(A-B)r/(1-B^2r^2)$ we derive the following results.

Theorem (1.1.6). Let $p(z)$ belong to the class $P(A,B)$ where $-1 \leq B < A \leq 1$. Then, for $|z| \leq r$, we have

$$\frac{1-Ar}{1-Br} \leq |p(z)| \leq \frac{1+Ar}{1+Br},$$

$$|\operatorname{Im} p(z)| \leq \frac{(A-B)r}{1-B^2r^2},$$

$$|\arg p(z)| \leq \tan^{-1}\left\{\frac{(A-B)r}{[(1-A^2r^2)(1-B^2r^2)]^{1/2}}\right\}.$$

Furthermore it is easily seen that $P(A,B) \subset P$ and for other suitable choice of the parameters A and B we obtain several well known interesting subclasses of P. We list some of them and state some relations existing between them (See[20]).

Definition (1.1.7).

(i) $P_\alpha \equiv P(1-2\alpha,-1) = \{p : \operatorname{Re} p(z) > \alpha,\ z \in E,\ 0 \leq \alpha < 1\}$

(ii) $P^\alpha \equiv P(\alpha, -\alpha) = \{p : |(p(z)-1)/(p(z)+1)| < \alpha,\ z \in E,\ 0 < \alpha \leq 1\}$

(iii) $P[\alpha] \equiv P(\alpha, 0) = \{p : |p(z)-1| < \alpha,\ z \in E,\ 0 \leq \alpha \leq 1\}$

(iv) $P(\alpha) \equiv P(1, \frac{1}{\alpha}-1) = \{p : |p(z)-\alpha| < \alpha,\ z \in E,\ \alpha > \frac{1}{2}\}$.

Remark (1.1.8). The following relations can be easily verified.

(i) $P(A,B) \subset P_{(1-A)/(1-B)}$ (ii) $P(A,B) \subset P(1/(1+B))$

(iii) $P(A,-1) \equiv P_{(1-A)/2}$ (iv) $P(1,B) \equiv P(1/(1+B))$.

We now proceed to introduce some well known subclasses of S which can be defined by simple geometrical properties. These classes can be characterised by simple analytic conditions involving the members of the class P. These properties and other basic properties can be found in [45] .

Definition (1.1.9)[45]. A function f in H is called univalently starlike if it maps E univalently onto a region starlike with respect to the origin, that is, the point w belongs to f(E) implies that the line segment joining the origin and the point w entirely belongs to f(E). We denote by S^* the class of starlike functions with respect to the origin. A function f belongs to S^* if and only if $zf'(z)/f(z)$ belongs to P. Geometrically this condition is equivalent to the fact that for a starlike function f, arg f(z) increases monotonically as z describes the circle $|z| = r$, $r < 1$.

Definition (1.1.10)[45]. A function $f \in H$ is starlike of order α, $0 \leq \alpha < 1$, if $\mathrm{Re}(zf'(z)/f(z)) > \alpha$, $z \in E$, and if for every $\epsilon > 0$ sufficiently small there is a z_0, $z_0 \in E$ for which $\mathrm{Re}\{z_0 f'(z_0)/f(z_0)\} < \alpha + \epsilon$. We denote this class by S^*_α; it is clear that $S^*_\alpha \subset S^*$ and when $\alpha = 0$ S^*_α coincides with S^*.

Theorem (1.1.11) [45]. Let $f \in S^*_\alpha$. Then for $|z| = r$, $r < 1$

(i) $\mathrm{Re}(zf'(z)/f(z)) \geq (1+(2\alpha-1)r)/(1+r)$,

(ii) $r/(1+r)^{2(1-\alpha)} \leq |f(z)| \leq r/(1-r)^{2(1-\alpha)}$.

These bounds are sharp for the function f_o defined by $f_o(z) = z(1-\epsilon z)^{-2(1-\alpha)}$, $|\epsilon| = 1$.

The class S^*_α was introduced by Robertson [45] and has been extensively investigated. This class is a special case of the more general subclass $S^*(A,B) = \{f \in H \; ; \; zf'(z)/f(z) \in P(A,B), \; z \in E\}$ of starlike functions introduced by Janowski [20]. We note $S^*(1-2\alpha, -1) \equiv S^*_\alpha$. The following special cases of $S^*(A,B)$, namely $S^*(\alpha, -\alpha)$ introduced by Padmanabhan [36], $S^*(1, \frac{1}{M} -1)$, $(M > \frac{1}{2})$ introduced by Janowski [19] and also studied by Singh [54], are of considerable interest.

Definition (1.1.2) [45]. A function f in H is convex univalent if it maps E univalently onto a convex region, that is, w_1, w_2 belongs to f(E) implies the line segment joining w_1 and w_2 belongs to f(E) or, equivalently, f(E) is starlike with respect to each of its points. We denote by K the class of convex functions in E. A function f is a member of K if and only if the function p defined by $p(z) = (1+zf''(z)/f'(z))$ $z \in E$ is a member of P. Geometrically this means that the function $w = f(re^{i\theta})$ maps each circle $|z| = r$, $r < 1$ onto a simple closed contour whose tangent rotates in the counter clockwise direction as $\arg z = \theta$ increases when z describes the circle $|z| = r$. It is easily seen that $f \in K$ if and only if $zf' \in S^*$.

Definition (1.1.13) [45] . A function $f \in H$ is convex of order β, $(0 \le \beta < 1)$, if $\mathrm{Re}(1+zf''(z)/f'(z)) > \beta$, $z \in E$, and if for every $\epsilon > 0$ sufficiently small there is a z_0, $z_0 \in E$ for which $\mathrm{Re}(1+z_0 f''(z_0))/f'(z_0) < \beta + \epsilon$. If K_β denotes this class it is clear that $K_\beta \subset K$ and $K_0 \equiv K$.

Theorem (1.1.14)[45]. Let $f \in K_\beta$. Then for $|z| = r$,
$((1+r)^{2\beta-1}-1)/(2\beta-1) \le |f(z)| \le (1-(1-r)^{2\beta-1})/(2\beta-1)$.
These bounds are sharp.

Theorem (1.1.15)[56] . Let $f \in K$. Then f satisfies $\mathrm{Re}(zf'(z)/f(z)) > 2^{-1}$ and $\mathrm{Re}(f(z)/z) > 2^{-1}$, $z \in E$. Consequently, for $|z| = r$,

$$\mathrm{Re}(zf'(z)/z)) \ge (1+r)^{-1} \quad \text{and}$$

$$|\arg(f(z)/z)| \le \text{arc Sin } r.$$

These bounds are attained by the function $z(1+z)^{-1}$.

Definition (1.1.16) [52]. A function $f \in H$ is said to be close-to-convex if there exists a function $g \in S^*$ and a real number ϕ, $|\phi| < \frac{\pi}{2}$ such that $\mathrm{Re}(e^{i\phi} zf'(z)/g(z)) > 0$, $z \in E$. We denote this class by C. The above inequality is equivalent to $\mathrm{Re}(e^{i\phi} f'(z)/h(z)) > 0$, $z \in E$ where $h \in K$. Also $f \in C$ if and only if $\int_{\theta_1}^{\theta_2} \mathrm{Re}(1+zf''(z)/f'(z))d\theta > -\pi$ whenever $0 \le \theta_1 < \theta_2 \le \theta_1+2\pi$ and $f'(z) \ne 0$ for $z = re^{i\theta}$ and $r < 1$.

Close-to-convex functions were introduced by Kaplan [23]. The above definition gives a normalised subclass of this class.

Theorem (1.1.17)[52]. $K \subset S^{*} \subset C \subset S$.

Definition (1.1.18) [25]. f is said to be close-to-convex of order α and type β, if there exists a function $g \in S^{*}_{\beta}$ and a real number ϕ, $|\phi| < \frac{\pi}{2}$ such that $\operatorname{Re}(e^{i\phi} zf'(z)/g(z)) > \alpha$, $z \in E$, $0 \leq \alpha < 1$. If $C(\alpha,\beta)$ denotes this class it is clear that $C(0,0) \equiv C$.

Mocanu [33] unified the classes of starlike functions and convex functions and introduced a fascinating class of functions known as α-convex functions. We denote this class by $M(\alpha)$.

Definition (1.1.19)[33]. A function $f \in H$ with $f(z)\, f'(z)/z \neq 0$ for $z \in E$ belongs to $M(\alpha)$ if and only if

$$\operatorname{Re}\left[\alpha\left(1+z\frac{f''(z)}{f'(z)}\right) + (1-\alpha)\, z\frac{f'(z)}{f(z)}\right] > 0,\ z \in E,$$

where α is any real number. For real α, α-convex functions are known to be starlike univalent [30].

This concept of unifying the existing classes **of univalent** functions attracted many researchers and several new classes have been introduced.

Definition (1.1.20)[8]. Let $N(\alpha)$ denote the class of functions

$f \in H$ satisfying the condition

$$\mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\right] > 0, \quad z \in E,$$

where $g \in S^*$ and α is a non-negative real number. It has been shown [8] that functions in $N(\alpha)$ are all close-to-convex.

In 1959, Sakaguchi [51] introduced a subclass of close-to-convex functions whose members are starlike with respect to symmetric points. We denote this class by S_s^*. The interesting geometrical feature possessed by these functions is revealed in the following definition.

<u>Definition (1.1.21)</u> [51]. Let f be analytic in E and suppose that for every $r \to 1$ $(r < 1)$ and every z_0 on $|z| = r$ the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction, that is

$$\mathrm{Re}[zf'(z)/(f(z)-f(-z_0))] > 0, \quad z = z_0, \quad |z_0| = r.$$

In this case f is called starlike with respect to symmetric points.

<u>Theorem (1.1.22)</u> [51]. A necessary and sufficient condition for $f \in H$ to be univalent and starlike with respect to symmetric points in E is that,

$$\mathrm{Re}\,[zf'(z)/(f(z)-f(-z))] > 0, \quad z \in E.$$

Remarks (1.1.23). The above condition implies that the vector $f(z)-f(-z)$ turns continuously in one direction as z describes each circle $|z| = r < 1$. It is also evident that the class S_s^* includes the classes of convex functions and odd functions starlike with respect to origin and is contained in the class of close-to-convex functions.

Theorem (1.1.24) [46]. Let the function $(1-t)f(z) + tf(-z)$ be subordinate to the univalent, analytic function $f(z) = z + \sum_2^\infty a_n z^n$ in E for an interval $0 \le t \le t_0$. Then f is starlike with respect to symmetric points in E.

Recently Das and Singh [11] extended the results of Sakaguchi to other classes of functions with respect to symmetric points in E.

Definition (1.1.25) [11]. A function $f \in H$ is said to be convex with respect to symmetric points in E if for every $r \to 1 (r < 1)$ and every z_0 on $|z| = r$

$$\text{Re}\,[z(zf'(z))'/(zf'(z)+z_0 f'(-z_0))] > 0 \text{ for } z = z_0, |z_0|=r.$$

We denote by K_s the class of convex functions with respect to symmetric points.

Theorem (1.1.26) [11]. A necessary and sufficient condition for $f \in H$ to be univalent and convex with respect to symmetric points in E is that,

$$\mathrm{Re}[\,(zf'(z))'/(f(z)-f(-z))'\,] > 0, \quad z \in E.$$

It immediately follows that $f \in K_s$ if and only if $zf'(z) \in S_s^*$.

<u>Definition (1.1.27)</u> [11]. A function $f \in H$ is said to be close-to-convex with respect to symmetric points if there exists a function $g \in S_s^*$ such that

$$\mathrm{Re}\,[zf'(z)/(g(z)-g(-z))] > 0, \quad z \in E.$$

We denote the class of close-to-convex functions with respect to symmetric points by C_s.

The class $C(\alpha,\beta)$ introduced by Libera (See Definition (1.1.18) induced Bharati [5] to introduce the following class.

<u>Definition (1.1.28)</u> [5] Let $L(A,B)$ denote the class of functions $f \in H$ satisfying the condition $z^2 f'(z)/g(z)h(z) \in P(A,B)$ where $g,h \in H$ and are subjected to the following conditions : (i) $g \in S_\beta^*$ and $h \in K$ (ii) $g, h \in K$ and (iii) g and h satisfy $\mathrm{Re}(g(z)/z) > 0$, $\mathrm{Re}(h(z)/z) > 0$ for $z \in E$. If $g \in S_\beta^*$ and $h(z) = z$ we note that $L(A,B) \subset C(0,\beta)$.

Next we list a few subclasses of H which generalise the classes of starlike and convex functions but are not necessarily univalent. These functions also possess nice geometrical properties.

Definition (1.1.29) [29]. For $k \geq 2$, let V_k denote the class of functions $f \in H$ that are locally univalent in E and map E onto a region with boundary rotation atmost $k\pi$. Analytically this is equivalent to the condition,

$$\int_0^{2\pi} |Re(1+re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta}))|\, d\theta \leq k\pi, \quad 0 \leq r < 1.$$

V_k is the class of functions of boundary rotation bounded by $k\pi$.

Remarks (1.1.30). If the boundary of a simply connected domain D has a continuous tangent at each of its points, the boundary rotation of D is the total variation, for a full turn, of the angle of direction of the tangent to D. V_2 coincides with the class of convex functions and Paatero [35] has shown that all functions in V_k, $k=2,3,4$ are also in S. However for $k > 4$ through counter examples it can be shown that there are functions in V_k that are not in S. The basic properties of V_k can be found in [29], [24] and [35].

Definition (1.1.31). [58]. Let U_k denote the class of functions $f \in H$ with $f(z)/z \neq 0$ in E and satisfying the condition

$$\int_0^{2\pi} | Re(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta}))|\,d\theta \leq k\pi, \quad 0 \leq r < 1.$$

The class U_k is known as the class of functions of argument rotation bounded by $k\pi$.

Remarks (1.1.32). Geometrically the above condition means that, the total variation of the angle which the radius vector $f(re^{i\theta})$ makes with the positive real axis is bounded by $k\pi$ as $z = re^{i\theta}$ describes the circle $|z| = r$ for each $r < 1$. We note that $U_2 \equiv S^*$, but for $k > 2$ the class U_k in general does not consist of univalent functions. From (1.1.29) and (1.1.31) it is clear that $f \in V_k$ if and only if $zf' \in U_k$.

Coonce and Ziegler [10] unified these classes and introduced the class of functions with bounded Mocanu Variation.

Definition (1.1.33)[10]. Let $MV[\alpha,k]$ ($\alpha \in \mathbb{R}$, $k \geq 2$) denote the class of functions with bounded Mocanu variation, that is, the subclass of functions $f \in H$ with $f'(z)f(z)/z \neq 0$ in E and satisfying the condition

$$\int_0^{2\pi} \left|\mathrm{Re}\left[(1-\alpha)re^{i\theta}f'(re^{i\theta})/f(re^{i\theta})+\alpha(1+re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta}))\right]\right| d\theta \leq k\pi,\ 0 \leq r < 1.$$

Remarks (1.1.34). A geometric interpretation of the above condition may be given as follows : Let C_r denote the image of the circle $|z| = r$ under the mapping $w = f(z)$. Then $\phi = \arg f(z)$ is the argument of the radius vector from the origin $w = 0$ to $w = f(z)$ and $\tau = \arg(izf'(z))$ is the argument of the tangent to C_r at $w = f(re^{i\theta})$ with direction determined by that of increasing θ. The angle $\psi=(1-\alpha)\phi+\alpha\tau$

is called the Mocanu angle. If we set $J(\alpha,f) = (1-\alpha)zf'(z)/f(z) + \alpha(1+zf''(z)/f'(z))$ it can be shown that $\frac{\partial \Psi}{\partial \theta} = \operatorname{Re} J(\alpha,f)$ (see [10]). Thus $f \in MV[\alpha,k]$ if and only if the total variation of the Mocanu angle Ψ is bounded by $k\pi$. It is easy to see that $MV[\alpha,2] \equiv M(\alpha)$ the class of α-convex functions. (See Definition (1.1.19)).

<u>Theorem (1.1.35)</u>. Jack's Lemma [17]. Let $d > 0$ and $w(z) : |z| < d \to \mathbb{C}$ be analytic, non-constant with $w(0) = 0$. If $|w|$ attains its maximum on the circle $|z| = r < d$ at z_0, then there exists $m \geq 1$ so that $z_0 w'(z_0) = m w(z_0)$.

Next we proceed to mention Goluzin's method for finding the solution of extremal problems for the classes of functions which have an integral representation.

Let E_g denote the class of regular functions having the representation $\int_{-\pi}^{\pi} g(z,t)\, d\alpha(t)$, where g is a fixed function of two variables z and t for $z \in E$ and for each t in $[-\pi, \pi]$ and $\alpha(t)$ runs through all possible nondecreasing functions in $-\pi \leq t \leq \pi$ subject to the condition $\int_{-\pi}^{\pi} d\alpha(t) = 1$. By a suitable variation of the function $\alpha(t)$ Goluzin has given two types of variations for functions in E_g. We state the two variational formulas proposed by Goluzin [15].

Theorem (1.1.36) [15]. Let $f \in E_g$, t_1, t_2 be given with $-\pi \leq t_1 < t_2 < \pi$ and let λ be any number in $[-1,1]$. Then there exists a real constant C independent of λ and t such that the functions f_* defined by

$$f_*(z) = f(z) + \lambda \int_{t_1}^{t_2} \frac{\partial g(z,t)}{\partial t} \, | \alpha(t) - C| \, dt, \quad z \in E,$$

are also in E_g.

Theorem (1.1.37) [15]. Let $f \in E_g$ and t_1 and t_2 with $-\pi \leq t_1 < t_2 < \pi$ be two jump points for the function $\alpha(t)$. Then there exists a number $\eta > 0$ such that for all λ in $(-\eta, \eta)$, the functions f_{**} defined by

$$f_{**}(z) = f(z) + \lambda[g(z,t_1) - g(z,t_2)]$$

are also in E_g.

Remarks (1.1.38). Goluzin [15] has solved certain extremal problems in the class P and S^* with the help of these variational formulas. This method has been adopted by Pinchuk ([40],[41]) to solve extreme problems for starlike and convex functions of order α and also for close-to-convex functions.

We next state some results concerning the Hardy class of some univalent functions and their derivatives.

Definition (1.1.39) [13]. For $p > 0$, a function $f \in H$ is said to belong to the Hardy class H^p if $\lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$ exists and is finite.

<u>Theorem (1.1.40)</u> [27]. If f is univalent then $f \in H^p$ for all $p < \frac{1}{2}$.

<u>Theorem (1.1.41)</u> [13]. If $f' \in H^p$ $(0 < p < 1)$ then $f \in H^{p/1-p}$.

<u>Theorem (1.1.42)</u> [13]. If $q \in P$ then $q \in H^p$ for all $p < 1$.

<u>Theorem (1.1.43)</u> [14]. (i) If $g \in S_\alpha^*$ then $g \in H^p$ for all $p < 1/2(1-\alpha)$ and $g' \in H^p$ for all $p < 1/(3-2\alpha)$.

(ii) If $g \in S_\alpha^*$ and $g(z) \neq g_t(z) = az(1-ze^{it})^{2\alpha-2}$ where t is real and a is complex then there exists an $\epsilon = \epsilon(g)$ such that $g \in H^{1/2(1-\alpha)+\epsilon}$ and $g' \in H^{1/(3-2\alpha)+\epsilon}$.

<u>Theorem (1.1.44)</u> [14]. (i) If $h \in K$ then $h' \in H^p$ for all $p < \frac{1}{2}$ and $h \in H^p$ for all $p < 1$.

(ii) If $h \in K$ and $h(z) \neq h_t(z) = \int_0^z g_t(w)w^{-1}dw$ where $g_t(z)$ is defined above then there exists an $\epsilon = \epsilon(h)$ such that $h' \in H^{\frac{1}{2}+\epsilon}$ and $h \in H^{1+\epsilon}$.

We require the integral representation of a hypergeometric function which plays a useful role in several branches of mathematics. The properties of the hypergeometric series and the associated function can be found in [64].

Definition (1.1.45) [64]. The series

$$F(\alpha,\beta,\gamma;z) = 1+\frac{\alpha\beta}{\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!} z^2 + \ldots$$

is called a hyper-geometric series. This series defines a function analytic in E. It is also analytic and single valued throughout the cut-plane having a cut from +1 to + ∞ along the real axis. The integral representation of the function is given by

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt$$

where $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$.

The conjecture of Pólya and Schoenberg [43] regarding the convolution of convex functions led to the study of convolution properties of several subclasses of S and discovery of new classes of analytic functions (See [57], [47], [48], [1]). We need the following classes.

Definition (1.1.46). Convolution of analytic functions. Let $f, g \in H$ where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then $(f * g)(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ is called the Hadamard product or convolution of f and g.

This product is commutative, associative and distributive with respect to addition in H. Other properties of convolution used in the thesis are stated at appropriate places.

Definition (1.1.47)[49]. A function f analytic in E normalised by $f(0) = 0$, $f'(0) \neq 0$ is called prestarlike of order α, $\alpha \leq 1$ if and only if

$$\mathrm{Re}(f(z)/zf'(0)) > 2^{-1},\ z \in E,\ \text{for } \alpha = 1,$$

$$z(1-z)^{-2(1-\alpha)} * f(z) \in S^*_\alpha,\ z \in E,\ \text{for } \alpha < 1;$$

where S^*_α denotes the class of all starlike functions of order α, $\alpha \leq 1$. We denote by L^*_α the class of prestarlike functions of order α.

Theorem (1.1.48) [49]. $L^*_0 \equiv K$, $L^*_{1/2} \equiv S^*_{1/2}$ and $L^*_\beta \subset L^*_\alpha$ whenever $\beta \leq \alpha \leq 1$.

Definition (1.1.49) [53]. Let $\Omega : H \to H$ be a continuous linear operator. Then Ω is called a convexity preserving operator if for each $f \in H$ the range of values of Ωf lies in the closed convex hull of the range of f.

Theorem (1.1.50) [49]. Let $f \in L^*_\alpha$, $g \in S^*_\alpha$ and $F \in H$. Then $\Omega F = (f * gF)/(f * g)$ is a convexity preserving convolution operator.

We next introduce a class of functions studied in [18] which unifies several well known classes.

Let $K_a(z) = z(1-z)^{-a}$, $a \in \mathbb{C}$ with Re $a > 0$, where a suitable branch is chosen so that $K_a \in H$. Let $f \in H$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, where $a_n \neq 0$ for all n and $\lim_{n \to \infty} |a_n|^{1/n} = 1$. Then, we denote by f^{-1} the unique well defined function in H for which $f^{-1} * f = K_1$.

<u>Definition (1.1.51)</u> [18]. Let $R_a^{\alpha}(a > 0,\ \alpha > 0)$ denote the class of functions $f \in H$ such that

$$\mathrm{Re}\left\{\alpha(a+1)\left[\frac{(K_{a+2} * f)(z)}{(K_{a+1} * f)(z)} - \frac{1}{2}\right] + (1-\alpha)\ a\left[\frac{(K_{a+1} * f)(z)}{(K_a * f)(z)} - \frac{a-1}{2a}\right]\right\} > 0$$

where $(K_a * f)(z)/z \neq 0$ and $(K_{a+1} * f)(z)/z \neq 0$ for $z \in E$.

<u>Definition (1.1.52)</u> [18]. Let $C_a (a > 0)$ denote the class of functions $f \in H$ such that

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[\frac{(K_{a+2} * f)(z)}{(K_{a+1} * f)(z)} - \frac{1}{2}\right] d\theta > \frac{-\pi}{a+1},$$

where $0 \leq \theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $z = re^{i\theta}$, $r < 1$ and $(K_{a+1} * f)(z)/z \neq 0$ for $z \in E$.

<u>Remarks (1.1.53)</u>. When $a = 1$, the class R_a^{α} coincides with the class $M(\alpha)$ of α-convex functions (See Definition (1.1.19)) and C_a reduces to the normalised class of close-to-convex

functions (see Definition (1.1.16)).

Theorem (1.1.54) [18]. We have the following relations.

(i) $R_a^1 \subset R_{a+1}^o$

(ii) $f \in R_a^o$ if and only if $K_a * f \in S^*_{(1-a)/2}$

(iii) $f \in R_a^1$ if and only if $K_{a+1} * f \in S^*_{(1-a)/2}$

(iv) $f \in R_a^1$ if and only if $K_a^{-1} * (K_{a+1} * f) \in R_a^o$.

It is easily seen that (iv) generalises the well known result, namely, $f \in K$ if and only if $zf' \in S^*$.

Theorem (1.1.55) [18]. $f \in C_a$ if and only if there exists a function $g \in R_a^1$ such that

$$\mathrm{Re}\left[\frac{(K_{a+1} * f)(z)}{(K_{a+1} * g)(z)}\right] > 0, \quad z \in E.$$

When a = 1, this reduces to a well known result due to Kaplan.

Theorem (1.1.56) [18]. We have $R_a^\beta \subset R_a^\alpha \subset C_a \subset C_b$ for $0 \leq \alpha \leq \beta$, $0 < b \leq a$, $a \geq 1$ and the functions in R_a^α and C_a are close-to-convex and hence univalent.

This theorem generalises the result $R_1^\alpha \subset R_1^o$ for $\alpha > 0$ [30] .

§ 1.2.

In this section we give a brief outline of the subject matter of the thesis. This thesis aims at defining and studying some new classes of functions analytic in the unit disc with special attention to the properties of univalence, representation and distortion theorems connected with them and their invariance under integral operators. In view of the fact that the study of univalent functions provides an interesting link between geometry and analysis, wherever possible, geometric interpretation of the analytic condition defining a new class has been furnished. The classes introduced in this thesis are either new subclasses of well known classes or they unify and generalise some of the existing classes whose definitions and important properties are outlined in the earlier section of this chapter.

The second chapter is devoted to the study of the class S_s^* of functions starlike with respect to symmetric points (See Definition (1.1.21)) and related classes of functions. We introduce and study some important subclasses of the class S_s^*.

We say a function $f \in H$ is a member of the class $S_s^*(A,B)$ $(-1 \leq B < A \leq 1)$ if and only if $2zf'(z)/(f(z)-f(-z))$ is subordinate to the function $(1+Az)/(1+Bz)$ in E. Clearly

$S_s^*(A,B) \subset S_s^*(1,-1) \equiv S_s^*$ and by giving suitable values for A and B several interesting subclasses of S_s^* can be realized as special cases. Certain properties like coefficient estimates, region of convexity and distortion theorems have been studied for the class $S_s^*(A,B)$.

We say a function $f \in H$ is strongly starlike of order α with respect to symmetric points if and only if $|\arg [zf'(z)/(f(z)-f(-z))]| < \alpha \frac{\pi}{2}$, $0 < \alpha \leq 1$, $z \in E$. If $S_s^*[\alpha]$ denotes this class it is easily seen that $S_s^*[\alpha] \subset S_s^*[1] \equiv S_s^*$. The effect of the integral transform given by $I(f) = (c+1)z^{-c} \int_0^z t^{c-1} f(t)dt$ where $c \in \mathbb{C}$ and $f \in S_s^*$ has been investigated. Representation and distortion theorems have also been derived for this class.

The classes of α-convex functions and α-close-to-convex functions with respect to symmetric points have also been introduced and studied. An extremal problem of a general nature has been solved for the class S_s^* using Goluzin's variational methods.

In the third chapter convolution properties have been used to define three new classes of analytic functions in E. Let

$$J(\alpha,f,a) = \alpha(a+1)\left[\frac{(K_{a+2} * f)(z)}{(K_{a+1} * f)(z)} - \frac{a}{a+1}\right] + (1-\alpha)a\left[\frac{(K_{a+1} * f)(z)}{(K_a * f)(z)} - \frac{a-1}{a}\right]$$

where $K_a(z) = z(1-z)^{-a}$, $a > 0$ and $\alpha \geq 0$.

We introduce the following classes of functions.

(i) $f \in MV[\alpha,k,a]$ $(\alpha > 0,\ k \geq 2)$ if and only if $f \in H$, $(K_a * f)(z)/z \neq 0$, $(K_{a+1} * f)(z)/z \neq 0$ for $z \in E$ and

$$\int_0^{2\pi} |\operatorname{Re} J(\alpha,f,a)|\, d\theta \leq k\pi, \quad z = re^{i\theta} \in E.$$

(ii) $f \in R_a^{\alpha}(\rho)$ $(\alpha > 0,\ \rho < 1)$ if and only if $f \in H$ and $\operatorname{Re} J(\alpha,f,a) > \rho$, $z \in E$.

(iii) $f \in C_a(\rho)$ $(\rho < 1)$ if and only if $f \in H$, $(K_{a+1} * f)(z)/z \neq 0$ for $z \in E$ and

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\{J(1,f,a) - \rho\}\, d\theta > -\pi, \quad z = re^{i\theta},\ 0 \leq r < 1,\ \theta_1 < \theta_2.$$

The class $MV[\alpha,k,a]$ generalises the well known class $MV[\alpha,k]$ (See Definition (1.1.33)) consisting of functions having bounded Mocanu variation introduced by Coonce and Ziegler and also unifies several other subclasses of H. An integral representation for $f \in MV[\alpha,k,a]$ is derived and a few other interesting properties of this class are investigated.

The classes $R_a^\alpha(\rho)$ and $C_a(\rho)$ generalise the classes R_a^α and C_a respectively introduced by Jankovics (See Definitions (1.1.51) and (1.1.52)). Representation and distortion theorems are obtained. Also properties of inclusion and univalence have been investigated.

The fourth chapter deals with the study of some radius of convexity problems associated with certain subclasses of H.

In the first section the following class of functions are introduced. Let f,g,h be functions belonging to the class H. The class R_k consists of functions f satisfying the condition $\int_0^{2\pi} |\operatorname{Re} z^2 f'(z)/g(z)h(z)|\, d\theta \leq k\pi$, $k \geq 2$, $z = re^{i\theta} \in E$. The radius of convexity of the class R_k is determined in each of the following cases.

(i) when g is starlike of order β and h is convex,

(ii) when g and h are both convex

(iii) when g and h satisfy $\operatorname{Re}[g(z)/z] > 0$, $\operatorname{Re}[h(z)/z] > 0$ for $z \in E$.

Coefficient estimates for $f \in R_k$, distortion theorems and the Hardy classes to which function in this class belong have also been obtained.

In the second section the radius of convexity is determined for functions defined by certain integral forms involving functions belonging to the class P.

Let $F(z) = \int_0^z (p(t))^{\beta} (f'(t))^{\gamma} dt$ where f(t) is convex and $\beta \geq 0$, $\gamma \geq 1$. The radius of convexity is determined in each of the following cases :

(i) $p \in P_{\alpha}$ (ii) $p \in P^{\alpha}$ (iii) $p \in P[\alpha]$ (iv) $p \in P(\alpha)$

(See Definition (1.1.7)).