Proceedings of the International Symposium on
Complex Analysis and Conformal Geometry (ISCACG 2013)
Editors: G. Bharali, D. Minda, S. Ponnusamy, and S. K. Sahoo
J. Analysis
Volume 21 (2013), 85–100

Rational functions with nodes

Masayo Fujimura and Masahiko Taniguchi

Dedicated to Professor Matti Vuorinen
on the occasion of his sixty-fifth birthday

Abstract. A natural kind of compactification of the virtual moduli spaces of rational functions of one complex variable is given. To describe the boundary points geometrically, the authors introduce the concept of rational functions with nodes, defined on partially crushed punctured Riemann spheres with nodes.

Keywords. Rational functions with nodes, compactification.

2010 MSC. Primary 30F60; Secondary 32G15, 37F30.

1. Introduction and main results

A *dynamical structure* of a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the Möbius conjugacy class of R. The *moduli space of rational functions* of degree d is the set of all dynamical structures of rational functions of degree d, and is denoted by M_d. For the details and backgrounds, see for instance, [3], [7], [8], and [9].

In this note, we introduce "rational functions with nodes" and the dynamical structures of them. Here, we also consider a natural kind of marking for such functions, and hence actually we discuss about marked rational functions with nodes. To give precise definitions of them, first we recall that a *∂-marked n-punctured Riemann sphere $\hat{S} = (\mathcal{D}(\hat{S}), N(\hat{S}))$ with nodes* is a pair of a family $\mathcal{D}(\hat{S}) = \{D_m\}_{m=1}^{M}$ of ∂-marked $n_m\,(\geq 3)$-punctured Riemann spheres D_m and the node set $N(\hat{S})$ consisting of J pairs $\{p_j, p'_j\}$ of punctures of different $D_{m(j)}$ and $D_{m'(j)}$, which satisfy that

$$\sum_{m=1}^{M} n_m = 2J + n, \qquad J = M - 1 \leq n - 3,$$

ISSN 0971-3611

and that the topological space $\hat{S}^*$ obtained from the disjoint union $|\mathcal{D}|(\hat{S}) = \sum_{m=1}^{M} D_m$ by filling and identifying all pairs of punctures in $N(\hat{S})$ is connected. Here, equipping $\hat{S}^*$ with a ∂-marking induced from that of $\hat{S}$, we call $\hat{S}^*$ the *realization* of $\hat{S}$. Cf., for instance, [1], [2], [5], and [6]. Also see Definition 1.4 below for the precise definition of ∂-marking.

A puncture of $\hat{S}$ is called *nodal* if it belongs to $N(\hat{S})$, and *non-nodal* otherwise. When we ignore ∂-marking, we call $\hat{S}$ simply an *n-punctured Riemann sphere with nodes*. In the sequel, we regard punctured Riemann spheres without nodes also as those with (no) nodes.

Now, the limit of rational functions may degenerate to the identity function on some components of the punctured Riemann sphere with nodes obtained as the limit of the punctured Riemann spheres where the functions are defined. To describe such phenomena precisely, we generalize the above definition as follows.

Definition 1.1 (Partial crush)**.** A *partial crush of level n* is an ordered pair $(\hat{S}, \hat{R})$ of an n-punctured Riemann sphere $\hat{S} = (\mathcal{D}(\hat{S}), N(\hat{S}))$ with nodes and a pair $\hat{R}$ of a family $\mathcal{D}(\hat{R}) = \{D'_i\}_{i=1}^{I}$ of n_i-punctured Riemann spheres D'_i and the node set $N(\hat{R})$ consisting of the pairs $\{p_r, p'_r\}$ of punctures of different $D'_{i(r)}$ and $D'_{i'(r)}$ such that

$$\mathcal{D}(\hat{R}) \subset \mathcal{D}(\hat{S}), \qquad N(\hat{R}) \subset N(\hat{S}),$$

which satisfies two additional conditions: Every pair in $N(\hat{S})$ contains a puncture of $\hat{R}$ and every component in $\mathcal{D}(\hat{S}) - \mathcal{D}(\hat{R})$ has at least two non-nodal punctures of $\hat{S}$.

We say that two partial crushes $(\hat{S}_1, \hat{R}_1)$ and $(\hat{S}_2, \hat{R}_2)$ of level n are *equivalent* if there is a homeomorphism $f : |\mathcal{D}|(\hat{S}_1) \to |\mathcal{D}|(\hat{S}_2)$ which preserves punctures and nodes and is a conformal map of $|\mathcal{D}|(\hat{R}_1)$ onto $|\mathcal{D}|(\hat{R}_2)$. A *partially crushed n-punctured Riemann sphere with nodes* is the equivalence class $[\hat{S}, \hat{R}]$ of a partial crush $(\hat{S}, \hat{R})$ of level n.

In the sequel, we abbreviate $[\hat{S}, \hat{R}]$ to $\hat{R}$ when it causes no confusion. Also, we call a component in $\mathcal{D}(\hat{S}) - \mathcal{D}(\hat{R})$ a *crushed component* of $\hat{R}$, though it is not a component in $\mathcal{D}(\hat{R})$.

Remark 1.2. Let $\overline{D'_i}^*$ be the closure of $D'_i \in \mathcal{D}(\hat{R})$ in $\hat{S}^*$. Then by definition, every component of $W = \hat{S}^* - \bigcup_{i=1}^{I} \overline{D'_i}^*$ is a punctured Riemann sphere belonging to $\mathcal{D}(\hat{S}) - \mathcal{D}(\hat{R})$. The second additional condition that every crushed component has at least two non-nodal punctures is crucial to give a natural kind of definition of "rational function with node". Also see the proof of Theorem 1.14.

Such an $[\hat{S}, \hat{R}]$ can be characterized by the crush data as follows. For every crushed component D, let $L(D)$ and $B(D)$ be the number of all non-nodal punctures of $\hat{S}$ on D and the set of all punctures of $\hat{R}$ in $N(\hat{S})$ paired with punctures on D, respectively. We call $B(D)$ a *singular bouquet* of punctures of $\hat{R}$, and $L(D)$ the *level* of the singular bouquet $B(D)$, which is also denoted by $L(B(D))$. Here note that every singular bouquet contains at most one puncture of D'_i for every D'_i in $\mathcal{D}(\hat{R})$. A puncture p in a singular bouquet B is called *singular*, and we call $L(B)$ also the *level* $L(p)$ of p.

Let $\{B_1, \cdots, B_N\}$ $(N = M - I)$ be the maximal set of the singular bouquets of $\hat{R}$. Then the set of all pairs $\{(B_\ell, L(B_\ell)) \mid \ell = 1, \cdots, N\}$ is called the *crush data of* $\hat{R}$. Let $\{q_1, \cdots, q_A\}$ be the set of all non-nodal non-singular punctures of $\hat{R}$, and we have

$$A + \sum_{\ell=1}^{N} L(B_\ell) = n.$$

We set $X(\hat{R}) = \{q_1, \cdots, q_A, B_1, \cdots, B_N\}$ and write $X(\hat{R})$ also as $\{q_r\}_{r=1}^{A+N}$. Further, set $L(q_r) = 1$ for every $r \leq A$.

Remark 1.3. The crush data of $\hat{R}$ depend on the choice of $\hat{S}$. But, possible choice of the set of the singular bouquets and their levels is finite in number. Also by the second additional condition in Definition 1.1, the level of every singular bouquet, or of every singular puncture, is not less than 2.

Definition 1.4 (∂-marking)**.** A *∂-marking* of a partially crushed n-punctured Riemann sphere $[\hat{S}, \hat{R}]$ with nodes is a surjection ι of $\{1, \cdots, n\}$ to the set $X(\hat{R})$ such that $\iota^{-1}(q_r)$ consist of $L(q_r)$ values for every $q_r \in X(\hat{R})$. If there are no crushed components, a ∂-marking is just an order of all non-nodal punctures.

We say that a partially crushed n-punctured Riemann sphere $[\hat{S}, \hat{R}]$, or simply $\hat{R}$, with nodes is *marked* if we equip $\hat{S}$ with a ∂-marking, which also canonically induces the ∂-marking of $\hat{R}$. In the sequel, a marked $\hat{R}$ is denoted by the same $\hat{R}$ unless we need the ∂-marking explicitly.

A ∂-marking of $\hat{R}$ can be considered also as an ordered set of disjoint subsets E_r consisting of $L(q_r)$ values in $\{1, \cdots, n\}$ for every $q_r \in X(\hat{R})$ which cover $\{1, \cdots, n\}$, and is always induced canonically from the ∂-marking of $\hat{S}$, where $\hat{R} = [\hat{S}, \hat{R}]$.

Definition 1.5 (Realization)**.** For a marked partially crushed n-punctured Riemann sphere $\hat{R}$ with nodes, let $\hat{R}^*$ be obtained from $|\mathcal{D}|(\hat{R})$ by filling and identifying every pair of punctures in $N(\hat{R})$ by one point, which is called a *non-singular* *node* of $\hat{R}^*$. Equipping $\hat{R}^*$ with the ∂-marking of $\hat{R}$, we call $\hat{R}^*$ the *realization* of $\hat{R}$.

Remark 1.6. $\hat{R}^*$ is not necessarily connected. If it is connected, then every singular bouquet consists of a single singular puncture.

We call every connected component of $\hat{R}^*$ an *ordinary part* of $\hat{R}$, while every D_i of $\mathcal{D}(\hat{R})$ an *ordinary component* of $\hat{R}$.

Definition 1.7 (Rational function with nodes)**.** A *rational function* $(\mathcal{F}, \hat{R})$ *with nodes of type* d is a family $\mathcal{F} = \{F_i\}_{i=1}^I$ of rational functions F_i on ordinary components D_i' of a partially crushed $(d+1)$-punctured Riemann sphere $\hat{R} = (\{D_i'\}_{i=1}^I, N(\hat{R}))$ with nodes satisfying the following conditions.

1. Every function F_i is not the identity and has its (not necessarily simple) fixed points only at punctures p of D_i' with multiplicity not greater than the level $L(p)$ of p.
2. (**Index formula at nodes**) Every nodal puncture of $\hat{R}$ is either a simple fixed point of, or not fixed by, the function in $\mathcal{F}$ corresponding to the puncture. The sum of the dynamical indices at the pair of punctures in the same node is 1.

Next, recall that the *generic locus* GM_d of the moduli space M_d is the sublocus corresponding to all rational functions of degree d with simple fixed points only, which are called *generic*. A marking of a generic rational function F is an order of $d+1$ fixed points of F. The set of the dynamical structures of all marked generic rational functions of degree d is called the *generic virtual moduli space* of degree d, and is denoted by GVM_d. Then as in [7], there are two canonical projections of GVM_d: Using the notations as in [5] and [10], the *Milnor projection* $\rho : GVM_d \to V\mathrm{Conf}(d+1, \widehat{\mathbb{C}})$ maps every point $[F]$ of GVM_d to the Möbius equivalence class of the ordered set of $d+1$ fixed points of F, and the *index decoration* $\Lambda : GVM_d \to (\mathbb{C}^*)^{d+1}$ maps $[F]$ to a $(d+1)$-dimensional vector

$$\Lambda([F]) = (\lambda_1([F]), \cdots, \lambda_{d+1}([F])),$$

where $\lambda_r([F])$ is the dynamical index of F at the r-th fixed point p_r for every r. The pair (ρ, Λ) of these projections gives a biholomorphic injection of GVM_d.

Now, we introduce a similar kind of marking and index decoration as above for rational functions with nodes.

Definition 1.8 (Reduced index decoration)**.** A *marking* of a rational function $(\mathcal{F}, \hat{R})$ with nodes is the ∂-marking ι of $\hat{R}$, or equivalently, the ordered set $\{E_r\}_{r=1}^{A+N}$. A marked rational function with nodes is denoted by a triple $(\mathcal{F}, \hat{R}, \iota)$, or more precisely, by $(\mathcal{F}, \hat{R}, \{E_r\})$.

Next, the *reduced index decoration* $\Lambda^\#$ for $(\mathcal{F}, \hat{R}, \iota)$ is defined at every puncture of $\hat{R}$ as follows: If $q \in X(\hat{R})$ is a non-singular puncture of $\hat{R}$, then the value of $\Lambda^\#$ at q is the index of F_q at q, where F_q is the function in $\mathcal{F}$ corresponding to q. If $q \in X(\hat{R})$ is a singular bouquet B, then for every singular puncture $p \in B$, the value of $\Lambda^\#$ at p is the coefficients $(c_1, \cdots c_L) \in \mathbb{C}^L$ of the principal part

$$\frac{c_1}{z} + \cdots + \frac{c_L}{z^L}$$

of the Laurent series expansion of $1/(z - F_p(z))$ at $p = 0$ with the global coordinate z of $\mathbb{C}$ and $L = L(B)$. Here, using the cyclic order of punctures induced from the ∂-marking on the component corresponding to p, we take two ordered punctures "adjacent" to p on the component, and send p and these two punctures to $0, 1, \infty$, respectively.

Attaching the vectors defined as above to all singular punctures, we obtain a point of $\mathbb{C}^{A+\sum_\ell \#(B_\ell)L(B_\ell)}$ as the reduced index decoration $\Lambda^\#$ for $(\mathcal{F}, \hat{R}, \iota)$, where $\#(B)$ is the cardinality of B.

Remark 1.9. In particular, c_1 in the above definition is the index of F_p at $p = 0$. (Cf. [4] and [10].) Here recall that the indices satisfy the *index formula*, i.e., the sum of all indices on an ordinary component is 1. Recall that c_ℓ except for c_1 depend on the choice of punctures normalized to be 1 and ∞.

Definition 1.10 (Dynamical structure)**.** We say that two marked rational functions $(\mathcal{F}, \hat{R}, \iota)$ and $(\mathcal{G}, \hat{R}', \iota')$ with nodes of type d are *marking-preserving Möbius conjugate* if

1. $[\hat{R}] = [\hat{R}']$ including the ∂-markings, i.e., there are Möbius transformations T_i for all i such that $D'_i = T_i(D_i)$ which preserve the ∂-markings and nodes including the order, where $\mathcal{D}(\hat{R}) = \{D_i\}_{i=1}^I$ and $\mathcal{D}(\hat{R}') = \{D'_i\}_{i=1}^I$, and
2. $\mathcal{F} = \{F_i\}_{i=1}^I$ is *marking-preserving Möbius conjugate to* $\mathcal{G} = \{G_i\}_{i=1}^I$ by the above $\{T_i\}_{i=1}^I$, i.e., $G_i = T_i \circ F_i \circ T_i^{-1}$ for every i.

A *dynamical structure* of a marked rational function $(\mathcal{F}, \hat{R}, \iota)$ with nodes is the marking-preserving Möbius conjugacy class of it, which is denoted by $[\mathcal{F}, \hat{R}, \iota]$. Note that the reduced index decoration $\Lambda^\#$ is well-defined for the class $[\mathcal{F}, \hat{R}, \iota]$.

The *virtual moduli space of rational functions with nodes of type* d is the set of the dynamical structures of all marked rational functions with nodes of type d, and is denoted by $\overline{VM}_d$. The sublocus of $\overline{VM}_d$ corresponding to usual rational functions of degree d is called the *virtual moduli space of degree* d and denoted by VM_d. Here, we regard that $\overline{VM}_d$ contains the "empty" point, i.e. the point $\infty = [\emptyset, \emptyset, \emptyset]$.

Finally, we introduce a standard kind of topology on $\overline{VM}_d$. For this purpose, we use the *reduced realization* $\hat{R}^{\#}$ of $\hat{R}$, which is obtained from $\hat{R}^*$ by filling all singular punctures in the same singular bouquet B by a single point q for every B. Every attached point q is called a *singular node*, which is distinguished from other ordinary points, even if B consists of a single singular puncture. We attach to $\hat{R}^{\#}$ the marking induced from that of $\hat{R}$, by replacing singular bouquets to singular nodes. Note that, even if $\hat{R}^*$ is disconnected, $\hat{R}^{\#}$ is always connected.

Definition 1.11 (Carathéodory convergence)**.** We say that points $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ converge to $[\mathcal{F}, \hat{R}, \iota]$ in $\overline{VM}_d$ *in the sense of Carathéodory* as $k \to \infty$ if there is an admissible sequence of continuous surjections $f_k : \hat{R}_k^{\#} \to \hat{R}^{\#}$ such that $F_{i,k} \circ f_k^{-1}$ converge to F_i spherically uniformly on $D_i - U$ for every ordinary component D_i of $\hat{R}$ and every neighborhood U of $N(\hat{R}) \bigcup S(\hat{R})$, where $F_{i,k}$ is a suitable rational function marking-preserving Möbius conjugate to the element in $\mathcal{F}_k$ defined on the ordinary component $D_{i,k}$ of $\hat{R}_k$ containing $f_k^{-1}(D_i)$, and $S(\hat{R})$ is the set of all singular nodes of $\hat{R}$.

Here, we say that a sequence $\{f_k : \hat{R}_k^{\#} \to \hat{R}^{\#}\}$ is *admissible* if

1. f_k^{-1} is a homeomorphism of D_i into an ordinary component of $\hat{R}_k$ for every k and every ordinary component D_i of $\hat{R}$,
2. $f_k^{-1}(p)$ is either a non-singular node of $\hat{R}_k^*$ or a simple closed curve on an ordinary component of $\hat{R}_k$ for every k and every non-singular node p of $\hat{R}^*$,
3. the relative boundary of $f_k^{-1}(\hat{R}^*)$ in $\hat{R}_k^*$ consists of a finite number of non-singular nodes and simple closed curves on ordinary components, and $f_k^{-1}(p)$ is a connected component of $\hat{R}_k^{\#} - f_k^{-1}(\hat{R}^*)$ for every k and every $p \in S(\hat{R})$,
4. the surjection $f_k^{\#} : X(\hat{R}_k) \to X(\hat{R})$ induced by f_k satisfies that $f_k^{\#} \circ \iota_k = \iota$ for every k, and
5. for every neighborhood V of the set of all punctures of $\hat{R}$ and every positive ϵ, f_k^{-1} is a $(1+\epsilon)$-quasiconformal map of $|\mathcal{D}|(\hat{R}) - V$ for every sufficiently large k.

Definition 1.12 (Strong convergence)**.** We say that $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ converge *strongly* to $[\mathcal{F}, \hat{R}, \iota]$ in $\overline{VM}_d$ as $k \to \infty$ if $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ converge to $[\mathcal{F}, \hat{R}, \iota]$ in the sense of Carathéodory as $k \to \infty$ and if $[\mathcal{F}, \hat{R}, \iota]$ is *maximal* in the sense that, if a subsequence of $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ converges to another $[\mathcal{G}, \hat{R}', \iota']$ in the sense of Carathéodory as $k \to \infty$, then $[\mathcal{G}, \hat{R}', \iota']$ is *subordinate* to $[\mathcal{F}, \hat{R}, \iota]$, i.e., we can find a continuous surjection $\phi : \hat{R}^{\#} \to (\hat{R}')^{\#}$ such that

1. ϕ^{-1} is a conformal map of D_i' onto an ordinary component of $\hat{R}$ for every ordinary component D_i' of $\hat{R}'$.

2. $\phi^{-1}(p)$ is a non-singular node of $\hat{R}^*$ for every non-singular node p of $(\hat{R}')^*$,
3. the relative boundary of $\phi^{-1}((\hat{R}')^*)$ in $\hat{R}^*$ consists of a finite number of non-singular nodes and $\phi^{-1}(p)$ is a connected component of $\hat{R}^\# - \phi^{-1}((\hat{R}')^*)$ for every $p \in S(\hat{R}')$, and
4. the surjection $\phi^\# : X(\hat{R}) \to X(\hat{R}')$ induced by ϕ satisfies that $\phi^\# \circ \iota = \iota'$.

Remark 1.13. Roughly speaking, if $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ in $\overline{VM}_d$ converge strongly to $[\mathcal{F}, \hat{R}, \iota]$ as $k \to \infty$ if and only if $\mathcal{F}_k$ converges to the identity function or not, respectively, exactly on the crushed components or on the ordinary ones of $\hat{R}$.

In general, there might be "superfluous" singular nodes of such $[\mathcal{G}, \hat{R}', \iota']$ as above.

By using strong convergence, we can introduce a topology on $\overline{VM}_d$, and conclude the following result. The proofs of all the assertions stated below will be given in the next section.

Theorem 1.14. *$\overline{VM}_d$ is a compact Hausdorff space.*

Definition 1.15 (Degree)**.** We call the closure of GVM_d in $\overline{VM}_d$ the *virtual moduli space of rational functions with nodes of degree* d, and is denoted by $\widehat{VM}_d$. We say that a marked rational function $(\mathcal{F}, \hat{R}, \iota)$ with nodes is of *degree* d for every $[\mathcal{F}, \hat{R}, \iota]$ in $\widehat{VM}_d$.

Theorem 1.16. *The virtual moduli space $\widehat{VM}_d$ of rational functions with nodes of degree d is compact, and the natural inclusion map of GVM_d into $\widehat{VM}_d$ can be extended to a continuous injection $\rho : VM_d \to \widehat{VM}_d$ with dense range.*

Here, we note the following fact.

Lemma 1.17. *Every marked rational function $(\mathcal{F}, \hat{R}, \iota)$ with nodes of type d such that the realization $\hat{R}^*$ is connected is of degree d.*

Example 1.18. Suppose that points $[P_k]$ in VM_d represent the classes of polynomials P_k of degree d, and $\rho([P_k])$ converge to $[\mathcal{F}, \hat{R}, \iota]$ in $\widehat{VM}_d$. Then, every F_m in $\mathcal{F}$ defined on an ordinary component D_m of $\hat{R}$ is either Möbius conjugate to a polynomial or to a constant.

In [3], we used $\mathcal{F}$ only to define the boundary point of such a sublocus of VM_d. Here, we also take the binding manner of them into account as $\hat{R}$. Hence the boundary of it constructed in this paper is larger than that in [3], and actually a finite branched cover of that. Also note that the realization of $\hat{R}$ is always connected in this case.

Proposition 1.19. *If $d \leq 5$ then $\overline{VM}_d = \widehat{VM}_d$.*

Finally, we will give an example (Example 2.3) in the next section which shows that $\overline{VM}_d - \widehat{VM}_d$ is non-empty for every $d \geq 6$.

Problem 1.20. For $d \geq 6$, find explicit conditions for rational function with nodes of type d to be of degree d.

2. Proofs

Proof of Theorem 1.14. It is easy to see that $\overline{VM}_d$ is Hausdorff and satisfies the second countability axiom, and hence it suffices to show sequential compactness of it. Thus, the next lemma implies the assertion. ∎

Lemma 2.1. *Let $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ be a sequence in $\overline{VM}_d$. Then we can find a subsequence which converges in $\overline{VM}_d$.*

Proof. We prove the assertion only for the case that all $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ belong to GVM_d, and hence in particular, $\mathcal{F}_k$ and $\hat{R}_k$ consist of a single function $F_{k,1}$ and a single component $D_{k,1}$, respectively, for every k. The general cases can be treated similarly. Also, taking a subsequence if necessary, we may assume that $[\hat{R}_k]$ converge to the point corresponding to a ∂-marked $(d+1)$-punctured Riemann sphere $\hat{S} = (\mathcal{D}(\hat{S}), N(\hat{S}))$ with nodes in the standard compactification $\widehat{V\mathrm{Conf}}(d+1, \widehat{\mathbb{C}})$ of $V\mathrm{Conf}(d+1, \widehat{\mathbb{C}})$. (Cf., for instance, [1], [2], and [5].)

Fix a component D_m in $\mathcal{D}(\hat{S})$, and let $\{p_r\}$ be the set of all punctures of D_m. Here, we may assume that $\{p_r\} \subset \mathbb{C}$. Furthermore, by taking a suitable representatives of $D_{k,1}$ and a subsequence if necessary, we may assume that j-th punctures $p_{k,j}$ of $D_{k,1}$ converge to one of p_r for every j. Let $Y(p_r)$ be the set of all j such that $p_{k,j}$ converge to p_r and $L'(p_r)$ is the number of such j for every p_r.

Again taking a subsequence if necessary, we may assume that, if $L'(p_r) = 1$, then the indices $\Lambda_k(j)$ of $F_{k,1}$ at $p_{k,j}$ converge to a value, say $\Lambda(p_r)$, in $\widehat{\mathbb{C}}$, where j is the unique element of $Y(p_r)$, and if $L'(p_r) > 1$, then

$$\sum_{j \in Y(p_r)} \frac{\Lambda_k(j)}{z - p_{k,j}} \quad \text{tend to} \quad \sum_{\ell=1}^{L'(p_r)} \frac{c_{\ell,r}}{(z-p_r)^\ell}$$

with respect to the global coordinate z on D_m, where some of $c_{\ell,r}$ might be ∞. We define a rational function F_{p_r} by setting

$$\frac{1}{z - F_{p_r}(z)} = \sum_{\ell=1}^{L'(p_r)} \frac{c_{\ell,r}}{(z-p_r)^\ell},$$

where we regard that F_{p_r} is the identity function $id.$ when some of $c_{\ell,r}$ are ∞.

If there exists either ∞ among $\Lambda(p_r)$ or $id.$ among F_{p_r} on D_m, then we classify the component D_m as a crushed one, and if not, as an ordinary one. Note that this classification does not depend on the representative of D_m. Hence we can define canonically a ∂-marked partially crashed $(d+1)$-punctured Riemann sphere $[\hat{S}, \hat{R}_0]$ with nodes, where $\hat{R}_0 = (\mathcal{D}(\hat{R}_0), N(\hat{R}_0))$ with $\mathcal{D}(\hat{R}_0) = \{D'_i\}$ consisting of all ordinary components in this classification. Then, every crushed component should contain at least two punctures of $\hat{S}$, for if not, no components adjacent to it can be ordinary. Also, it is easy to construct an admissible family of continuous surjections $f_k : \hat{R}_k^{\#} = \hat{R}_k \to \hat{R}_0^{\#}$ such that $F_{k,1} \circ f_k^{-1}$ converges to a rational function $F_{D'_i}$ defined by

$$\frac{1}{z - F_{D'_i}(z)} = \sum_{L'(p_r)=1} \frac{\Lambda(p_r)}{z - p_r} + \sum_{L'(p_r)>1} \frac{1}{z - F_{p_r}(z)}$$

locally uniformly on every D'_i, where $\{p_r\}$ is as above with $D_m = D'_i$.

Finally, let $\mathcal{F}$ be the set of all these $F_{D'_i}$ on ordinary components D'_i of $\hat{R}_0$. Then it is easy to see that $[\mathcal{F}_k, \hat{R}_k, \iota_k]$ converge strongly to $[\mathcal{F}, \hat{R}_0, \iota]$ with naturally induced marking ι, and we have the assertion. ■

Next, we note the following fact.

Lemma 2.2. *The natural inclusion map of GVM_d into $\widehat{VM}_d$ can be extended canonically to a continuous injection $\rho : VM_d \to \widehat{VM}_d$.*

Proof. Every point $[F]$ in $VM_d - GVM_d$ corresponds to such an $\hat{R} = (\mathcal{D}(\hat{R}), \emptyset)$ that $\mathcal{D}(\hat{R})$ consists of a single marked punctured Riemann sphere, say D, and the ∂-marking ι is determined from F. Here, multiple fixed points of F correspond to singular punctures of $\hat{R}$ and every singular bouquet consists of a single singular puncture. The level of every singular puncture p equals the multiplicity of the fixed point of F at p.

Now, we set $\rho([F])$ to be the point $[\{F\}, \hat{R}, \iota]$ of $\overline{VM}_d$. Then from the construction, the realization $\hat{R}^*$ is connected. Hence Lemma 1.17, which is proved next but independently, implies that $\rho([F])$ is contained in $\widehat{VM}_d$. From the definition of topology, it is easy to conclude that the map

$$\rho : VM_d \to \widehat{VM}_d$$

defined above is a continuous injection. ■

Proof of Theorem 1.16. Denseness of $\rho(VM_d)$ in $\widehat{VM}_d$ is trivial from the definition of $\widehat{VM}_d$, and we conclude Theorem 1.16 by Theorem 1.14 and Lemma 2.2. ■

Proof of Lemma 1.17. Suppose that a marked rational function $(\mathcal{F}, \hat{R}, \iota)$ with nodes of type d admits a connected realization $\hat{R}^*$. Then, every singular bouquet B_ℓ consists of a single singular puncture, say q_ℓ.

Take a representative of the component containing q_ℓ such that $q_\ell = 0$ and let the reduced index decoration of $F \in \mathcal{F}$ corresponding to q_ℓ at 0 be $(c_1, \cdots c_{L_\ell})$, where L_ℓ is the level of q_ℓ.

For every $\epsilon = \{\epsilon_\nu\}$ with mutually distinct ϵ_ν sufficiently near 0, let $F_{q_\ell,\epsilon} = F_{q_\ell,\{\epsilon_\nu\}}(z)$ be defined by

$$\frac{1}{z - F_{q_\ell,\epsilon}(z)} = \sum_{\nu=1}^{L_\ell} \frac{\lambda_\nu}{z - \epsilon_\nu} \left(= \frac{A_{L_\ell-1} z^{L_\ell-1} + \cdots + A_0}{\prod (z - \epsilon_\nu)} \right),$$

where λ_ν are non-zero and A_ν depend on λ_ν and ϵ_ν. More precisely, $F_{q_\ell,\epsilon}(z)$ can be written as

$$\begin{aligned} \frac{1}{z - F_{q_\ell,\epsilon}(z)} &= \frac{\sum_{\nu=1}^{L_\ell} \lambda_\nu (z - \epsilon_1) \cdots (z \breve{-} \epsilon_\nu) \cdots (z - \epsilon_{L_\ell})}{\prod_{\nu=1}^{L_\ell} (z - \epsilon_\nu)} \\ &= \frac{\sum_{\nu=1}^{L_\ell} \lambda_\nu (z^{L_\ell-1} - \sigma_\nu^{(1)} z^{L_\ell-2} + \sigma_\nu^{(2)} z^{L_\ell-3} - \cdots + (-1)^{L_\ell-1} \sigma_\nu^{(L_\ell-1)})}{\prod_{\nu=1}^{L_\ell} (z - \epsilon_\nu)}, \end{aligned}$$

with the j-th elementary symmetric functions $\sigma_\nu^{(j)}$ of $\epsilon_1, \cdots, \breve{\epsilon_\nu}, \cdots, \epsilon_{L_\ell}$. Here, $(z \breve{-} \epsilon_\nu)$ and $\breve{\epsilon_\nu}$ mean the deletion of $(z - \epsilon_\nu)$ and ϵ_ν, respectively. Hence A_ν can be expressed as

$$\begin{pmatrix} A_{L_\ell-1} \\ -A_{L_\ell-2} \\ \vdots \\ (-1)^{L_\ell-1} A_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_{L_\ell}^{(1)} \\ \vdots & & & \vdots \\ \sigma_1^{(L_\ell-1)} & \sigma_2^{(L_\ell-1)} & \cdots & \sigma_{L_\ell}^{(L_\ell-1)} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{L_\ell} \end{pmatrix}.$$

Let M be the $L_\ell \times L_\ell$ matrix in the right hand side. Then we can show that

$$\det M = \prod_{\mu<\nu} (\epsilon_\mu - \epsilon_\nu).$$

Hence, for mutually distinct $\{\epsilon_\nu\}$, also $\{A_\nu\}$ determines $\{\lambda_\nu\}$ uniquely. Actually, the inverse matrix of M is

$$\left(\frac{(-1)^{j-1} \epsilon_k^{L_\ell-j}}{\Delta_k} \right) \qquad \text{with} \quad \Delta_k = \prod_{\nu \neq k} (\epsilon_k - \epsilon_\nu).$$

Now, take mutually distinct $\{\epsilon_\nu\}$ arbitrarily near 0 and fix suitable values $\{(-1)^{\nu-1}A_{L_\ell-\nu}\}$ arbitrarily near $\{c_\nu\}$ so that the solution $\{\lambda_\nu\}$ determined from them consists of non-zero values only. Let $F_{q_\ell,\epsilon}$ be defined as above with these ϵ_ν and λ_ν for every q_ℓ.

For every component D of $\hat{R}$, let $\{p_{D,r}\}$ and $\{q_{D,\ell}\}$ be the sets of all non-singular punctures and of all singular ones, respectively, of D. Then after taking suitable conjugates of $F_{q_{D,\ell},\epsilon}$ if necessary, we have a generic rational function $F_{D,\epsilon}$ on D defined by

$$\frac{1}{z-F_{D,\epsilon}(z)}=\sum_{\{p_{D,r}\}}\frac{\lambda'_{p_{D,r}}}{z-p_{D,r}}+\sum_{\{q_{D,\ell}\}}\frac{1}{z-F_{q_{D,\ell},\epsilon}(z)}.$$

Here $\lambda'_{p_{D,r}}$ are non-zero values arbitrarily near to the indices $\lambda_{p_{D,r}}$ of the corresponding element of $\mathcal{F}$ at $p_{D,r}$, which satisfy the index relation:

$$\sum_{\{p_{D,r}\}}\lambda'_{p_{D,r}}+\sum_{\{q_{D,\ell}\}}A_{q_{D,\ell},L_{\ell-1}}=1,$$

where $A_{q_{D,\ell},L_\ell-1}$ is $A_{L_\ell-1}$ for $q_{D,\ell}$ as above for every $q_{D,\ell}$. Set $\mathcal{F}_\epsilon=\{F_{D,\epsilon}\}$, where D moves all components of $\hat{R}$, and we obtain points $[\mathcal{F}_\epsilon,\hat{R},\iota]$ arbitrarily near to the given $[\mathcal{F},\hat{R},\iota]$ in $\overline{VM}_d$.

Now by a standard surgery by reopening non-singular nodes, we can approximate such $[\mathcal{F}_\epsilon,\hat{R},\iota]$ arbitrarily by points in GVM_d, which implies that $(\mathcal{F},\hat{R},\iota)$ is of degree d, and we have proved Lemma 1.17. ■

Proof of Proposition 1.19. When $d\le 4$, it is easy to see that there are no partially crushed $(d+1)$-punctured Riemann sphere $\hat{R}$ with nodes such that the realization of $\hat{R}$ is disconnected. Thus Proposition 1.19 follows from Lemma 1.17.

Suppose that $d=5$, and let $\hat{R}$ be a partially crushed 6 punctured Riemann sphere with nodes such that the realization of $\hat{R}$ is disconnected. Then there is essentially only one possibility for $\hat{R}$, namely, $\hat{R}$ is obtained from 6 punctured Riemann sphere $\hat{S}=(\{D_1,D_2,D_3\},N(\hat{S}))$ with two nodes, which connect D_3 with D_1 and with D_2, by crushing D_3. Hence, the single singular bouquet B of $\hat{R}$ consists of two punctures of D_1 and D_2, and the levels of them are 2. We may assume that $D_1=D_2=\mathbb{C}-\{0,1\}$, and 0 corresponds to the singular punctures.

Let $[\mathcal{F},\hat{R},\iota]$ be any point in $\overline{VM}_5$, and $(\lambda_1^j,\lambda_2^j;(c_1^j,c_2^j))$ be the reduced index decoration on D_j corresponding to $\mathcal{F}$ for each j. Then c_1^j is determined from λ_1^j and λ_2^j by the index relation on D_j. For every k, we consider a generic rational function F_k of degree 5 defined by

$$\frac{1}{z-F_k(z)} = \frac{\tilde\lambda^1_{k,1}}{z+2} + \frac{\tilde\lambda^1_{k,2}}{z+2-\epsilon_{k,1}} + \frac{\tilde\lambda^2_{k,1}}{z-2} + \frac{\tilde\lambda^2_{k,2}}{z-2-\epsilon_{k,2}} + \frac{\tilde c_{k,1}}{z} + \frac{\tilde c_{k,2}}{z-1},$$

where the fixed points $0, (-1)^j\, 2, (-1)^j\, 2 + \epsilon_{k,j}$ correspond to punctures $0, \infty, 1$ of D_j for each j and every k, $\tilde\lambda^j_{k,\nu}$ and $\tilde c_{k,\ell}$ are non-zero and satisfy that

$$\sum_{j,\nu} \tilde\lambda^j_{k,\nu} + \sum_{\ell} \tilde c_{k,\ell} = 1,$$

for every k, and $\epsilon_{k,j}$ converge to 0 for each j and $\tilde\lambda^j_{k,\nu}$ tend to λ^j_ν for every j and ν as $k \to \infty$. Since D_3 is crushed, we should choose $\tilde c_{k,\ell}$ so that they tend to ∞.

Now, we set

$$\epsilon_{k,j} = \frac{a^j_k}{k^2}$$

with bounded non-zero a^j_k for every k and j, which are determined below. Take the conformal embeddings of $\mathbb{C} - \{0, 1, \pm 2, (-1)^j 2 + \epsilon_{k,j}\}$ into $\mathbb{C} - \{0, 1\}$ which fix 0 and send $(-1)^j 2$ and $(-1)^j 2 + \epsilon_{k,j}$ to ∞ and 1, respectively, and hence are the Möbius transformations

$$S_{k,j}(z) = \mu_{k,j}\, \frac{z}{z-(-1)^j 2} \qquad \text{with} \quad \mu_{k,j} = \frac{\epsilon_{k,j}}{(-1)^j 2 + \epsilon_{k,j}} \approx \frac{\epsilon_{k,j}}{(-1)^j\, 2}.$$

(Here and in the sequel, $a_k \approx b_k$ means $\lim_{k\to\infty} a_k/b_k = 1$.) Note that

$$\tilde c_{k,1} + \tilde c_{k,2} \approx c^1_1 + c^2_1 - 1 = 1 - \sum_{j,\nu} \lambda^j_\nu, \quad \text{and} \quad \tilde c_{k,1} S_{k,j}(1) \approx -c^j_2.$$

Now, if c^j_2 is non-zero, then we set $b^j_k = -c^j_2$, and if not, then we set $b^j_k = 1/k$, for every k and each j. Then we can find bounded non-zero values a^j_k which satisfy the equation

$$\frac{b_{k,1}}{S_{k,1}(1)} = \frac{b_{k,2}}{S_{k,2}(1)}$$

which we take as $\tilde c_{k,1}$ for every k. Here note that, if $b^2_k = o(b^1_k)$, for instance, then we can take such a^j_k that $a^2_k = o(a^1_k)$ and hence $\epsilon_{k,2} = o(\epsilon_{k,1})$.

These F_k determine the points in GVM_d with the marking induced from above. By construction, $\tilde c_{k,\ell}$ tend to ∞, and hence we can see that they converge to $[\hat F, \hat R, \iota]$ as $k \to \infty$. Thus we conclude the assertion. ∎

Finally, we show that Proposition 1.19 is best possible. Actually, $\overline{VM_d} - \widehat{VM}_d$ is non-empty for every $d \geq 6$.

Example 2.3. For the sake of simplicity, we consider the case $d = 6$ only, for the other cases can be treated by the same arguments. Let $[\mathcal{G}, \hat{R}, \iota]$ be a point in $\overline{VM}_6$, where $\hat{R} = (\{D_1, D_2\}, N(\hat{R}))$ is as in the proof of Proposition 1.19, i.e., $\hat{R}^*$ is disconnected and D_j $(j = 1, 2)$ are $\mathbb{C} - \{0, 1\}$, whose punctures at 0 are in the same singular bouquet, which is of level 3 in this case. We set $\mathcal{G} = \{G_1, G_2\}$, where G_j are defined by

$$\frac{1}{z - G_j(z)} = \frac{2}{z-1} - \frac{1}{z} + \frac{(-1)^j}{z^2} + \frac{1}{z^3}.$$

Then $[\mathcal{G}, \hat{R}, \iota] \in \overline{VM}_6 - \widehat{VM}_6$.

Indeed, if not, then it is the limit of a suitable sequence of points in VGM_6 determined by generic rational functions F_k of degree 6. By taking a Möbius conjugate if necessary, we may assume that such F_k are given by

$$\begin{aligned}\frac{1}{z - F_k(z)} \quad &= \quad \frac{\eta_{k,1}}{z+2} + \frac{\kappa_{k,1}}{z+2-\epsilon_{k,1}} + \frac{\eta_{k,2}}{z-2} + \frac{\kappa_{k,2}}{z-2-\epsilon_{k,2}} \\ &\quad + \frac{\lambda_{k,1}}{z} + \frac{\lambda_{k,2}}{z-\delta_k} + \frac{\lambda_{k,3}}{z-\delta'_k},\end{aligned}$$

where

$$\sum_j (\eta_{k,j} + \kappa_{k,j}) + \sum_\nu \lambda_{k,\nu} = 1,$$

$\kappa_{k,j}$ and $\epsilon_{k,j}$ are non-zero and tend to 0, while $\eta_{k,j}$ tend to 2, as $k \to \infty$ for each j, and $\lambda_{k,\nu}$ tend to ∞ as $k \to \infty$ for some ν. Also, δ_k and δ'_k are mutually distinct, equal none of $\{0, \pm 2, (-1)^j\, 2 + \epsilon_{k,j}\}$, and may be assumed to converge to finite values, say a and a', respectively. Note that a, a' may belong to $\{0, \pm 2\}$.

Now, we may assume that the marking-preserving conformal embeddings of

$$\mathbb{C} - \{0, \pm 2, \delta_k, \delta'_k, (-1)^j\, 2 + \epsilon_{k,j}\}$$

into $\mathbb{C} - \{0, 1\}$ fix 0 and send $(-1)^j\, 2$ and$(-1)^j\, 2 + \epsilon_{k,j}$ to ∞ and 1, respectively, and hence are again given by Möbius transformations $S_{k,j}(z)$ defined in the proof of Proposition 1.19.

From the assumption, we may assume without loss of generality, that $S_{k,j}(\delta_k)$ and $S_{k,j}(\delta'_k)$ converge to 0 for each j, $\sum_\nu \lambda_{k,\nu} = -3$ for every k, and $G_{k,j}$ defined by

$$\begin{aligned}\frac{1}{z - G_{k,j}(z)} \quad &= \quad \frac{\lambda_{k,1}}{z} + \frac{\lambda_{k,2}}{z - S_{k,j}(\delta_k)} + \frac{\lambda_{k,3}}{z - S_{k,j}(\delta'_k)} \\ &\quad + \frac{2}{z-1} + \frac{2}{z - S_{k,j}((-1)^{3-j}\, 2 + \epsilon_{k,3-j})}\end{aligned}$$

converge to G_j for each j as $k \to \infty$. Set

$$\frac{\lambda_{k,1}}{z} + \frac{\lambda_{k,2}}{z - S_{k,j}(\delta_k)} + \frac{\lambda_{k,3}}{z - S_{k,j}(\delta'_k)} = \frac{-3z^2 + b_{k,j}z + a_{k,j}}{z(z - S_{k,j}(\delta_k))(z - S_{k,j}(\delta'_k))},$$

and write $S_{k,j}(\delta_k)$ and $S_{k,j}(\delta'_k)$ simply as $S_{k,j}$ and $S'_{k,j}$. Then simple computations show that

$$\begin{aligned}
\lambda_{k,1} &= \frac{a_{k,j}}{S_{k,j}S'_{k,j}}, \\
\lambda_{k,2} &= \frac{-a_{k,j} - b_{k,j}S_{k,j} + 3S_{k,j}^2}{S_{k,j}(S'_{k,j} - S_{k,j})}, \\
\lambda_{k,3} &= \frac{a_{k,j} + b_{k,j}S'_{k,j} - 3(S'_{k,j})^2}{S'_{k,j}(S'_{k,j} - S_{k,j})}.
\end{aligned}$$

Here recall that $(a_{k,j}, b_{k,j})$ converge to $(1, (-1)^j)$ for each j.

Now, by the first equation, we have

$$\frac{a_{k,1}}{S_{k,1}S'_{k,1}} = \frac{a_{k,2}}{S_{k,2}S'_{k,2}},$$

or more precisely,

$$\frac{\mu_{k,1}^2}{a_{k,1}(\delta_k + 2)(\delta'_k + 2)} = \frac{\mu_{k,2}^2}{a_{k,2}(\delta_k - 2)(\delta'_k - 2)}.$$

Hence

$$\begin{aligned}
\frac{S_{k,1}(S'_{k,1} - S_{k,1})}{a_{k,1}} &= \frac{\mu_{k,1}^2}{a_{k,1}} \frac{\delta_k}{\delta_k + 2} \frac{2(\delta'_k - \delta_k)}{(\delta_k + 2)(\delta'_k + 2)} \\
&= \frac{2 - \delta_k}{2 + \delta_k} \frac{\mu_{k,2}^2}{a_{k,2}} \frac{\delta_k}{\delta_k - 2} \frac{-2(\delta'_k - \delta_k)}{(\delta_k - 2)(\delta'_k - 2)} \\
&= \frac{2 - \delta_k}{2 + \delta_k} \frac{S_{k,2}(S'_{k,2} - S_{k,2})}{a_{k,2}}.
\end{aligned}$$

Here, a rough estimate of the equation for $\lambda_{k,2}$ shows that

$$\lambda_{k,2} \approx \frac{-a_{k,1}}{S_{k,1}(S'_{k,1} - S_{k,1})} \approx \frac{-a_{k,2}}{S_{k,2}(S'_{k,2} - S_{k,2})}.$$

Thus

$$\frac{2 + \delta_k}{2 - \delta_k} \approx 1, \quad \text{i.e.,} \quad \delta_k \to 0.$$

Since $a_{k,j} \to 1$ and $\delta_k \neq 0$, we conclude that

$$\begin{aligned} & \frac{-a_{k,1}}{S_{k,1}(S'_{k,1} - S_{k,1})} - \frac{-a_{k,2}}{S_{k,2}(S'_{k,2} - S_{k,2})} \\ \approx \quad & \frac{-2\delta_k}{2 - \delta_k} \frac{-a_{k,1}}{S_{k,1}(S'_{k,1} - S_{k,1})}, \end{aligned}$$

which is non-zero and not

$$\frac{o(\delta_k)}{S_{k,1}(S'_{k,1} - S_{k,1})}$$

as $k \to \infty$.

On the other hand, since

$$S_{k,j} = O(\epsilon_{k,j}\delta_k) = o(\delta_k),$$

we should have

$$\begin{aligned} 0 \quad = \quad & \frac{-a_{k,1} - b_{k,1}S_{k,1} + 3S_{k,1}^2}{S_{k,1}(S'_{k,1} - S_{k,1})} - \frac{-a_{k,2} - b_{k,2}S_{k,2} + 3S_{k,2}^2}{S_{k,2}(S'_{k,2} - S_{k,2})} \\ = \quad & \frac{(-a_{k,1} + o(\delta_k))}{S_{k,1}(S'_{k,1} - S_{k,1})} - \frac{(-a_{k,2} + o(\delta_k))}{S_{k,j}(S'_{k,2} - S_{k,2})} \\ = \quad & \frac{-a_{k,1}}{S_{k,1}(S'_{k,1} - S_{k,1})} - \frac{-a_{k,2}}{S_{k,j}(S'_{k,2} - S_{k,2})} + \frac{o(\delta_k)}{S_{k,1}(S'_{k,1} - S_{k,1})}, \end{aligned}$$

which is a contradiction.

Remark 2.4. The condition that the "last" components in the reduced index decorations at 0 are non-zero is crucial.

Also note that, even if we consider to approximate $[\mathcal{G}, \hat{R}, \iota]$ by points in $VM_d - GVM_d$ corresponding to the condition that $\delta_k = \delta'_k = 0$, we still have a contradiction more easily. Actually, if the approximating functions H_k are defined by

$$\begin{aligned} \frac{1}{z - H_k(z)} \quad = \quad & \frac{\eta_{k,1}}{z+2} + \frac{\kappa_{k,1}}{z + 2 - \epsilon_{k,1}} + \frac{\eta_{k,2}}{z-2} + \frac{\kappa_{k,2}}{z - 2 - \epsilon_{k,2}} \\ & + \frac{\lambda_{k,1}}{z} + \frac{\lambda_{k,2}}{z^2} + \frac{\lambda_{k,3}}{z^3}, \end{aligned}$$

where $\sum_j (\eta_{k,j} + \kappa_{k,j}) + \lambda_{k,1} = 1$ and $\eta_{k,j}$ and $\kappa_{k,j}$ tend to 2 and 0, respectively, as $k \to \infty$, then the corresponding $G_{k,j}$ as before can be defined by

$$\begin{aligned} \frac{1}{z - G_{k,j}(z)} \quad = \quad & -\frac{3}{z} - \frac{\epsilon_{k,j}(\lambda_{k,2} + (-1)^j 2\lambda_{k,3})}{4z^2} + \frac{\epsilon_{k,j}^2 \lambda_{k,3}}{4z^3} \\ & + \frac{2}{z-1} + \frac{2}{z - S_{k,j}((-1)^{2-j}\, 2 + \epsilon_{k,2-j})}, \end{aligned}$$

which should converge to G_j for each j as $k \to \infty$. But this is again impossible.

Acknowledgements. The first author was partially supported by Grants-in-Aid for Scientific Research (C) (Grant No. 22540240). The second author was partially supported by Grants-in-Aid for Scientific Research (C) (Grant No. 23540202) and Grant-in-Aids for Scientific Research (B) (Grant No. 25287021). The authors would like to express heartfelt thanks to the referee for his/her valuable comments.

References

[1] L. Bers, On spaces of Riemann surfaces with nodes, *Bull. Amer. Math. Soc.*, **80** (1974), 1219–1222.

[2] X. Buff, J. Fehrenbach, P. Lochak, L. Schneps, and P. Vogel, *Moduli spaces of curves, mapping class groups and field theory*, SMF/AMS Texts and Monographs, **9**, Amer. Math. Soc., 2003.

[3] M. Fujimura and M. Taniguchi, A compactification of the moduli space of polynomials, *Proc. Amer. Math. Soc.*, **136** (2008), 3601–3609.

[4] M. Fujimura and M. Taniguchi, Stratification and coordinate systems for the moduli space of rational functions, *Conform. Geom. Dyn.*, **14** (2010), 141–153.

[5] M. Funahashi and M. Taniguchi, The cross-ratio compactification of the configuration space of ordered points on $\widehat{\mathbb{C}}$, *Acta Math. Sinica (English)*, **28** (2012), 2129–2138.

[6] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer-Verlag, Tokyo, 1992.

[7] J. Milnor, *Dynamics in one complex variable,* Third edition, Princeton University Press, 2006.

[8] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic Dynamics*, Cambridge Univ. Press, 2000.

[9] J. Silverman, The space of rational maps on $\mathbb{P}^1$, *Duke Math. J.*, **94** (1998), 41–77.

[10] M. Taniguchi, On spaces of rational functions, In: K. Matsuzaki and T. Sugawa (Eds.) *Topics in Finite or Infinite Dimensional Complex Analysis*, Tohoku Univ. Press, 63–74, 2013.

Masayo Fujimura
Address:
Department of Mathematics
National Defense Academy of Japan
239-0811, Yokosuka, Japan
E-mail: `masayo@nda.ac.jp`

Masahiko Taniguchi
Address:
Department of Mathematics
Nara Women's University
630-8506, Nara, Japan
E-mail: `tanig@cc.nara-wu.ac.jp`