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Operators on locally Hilbert space

Dinesh J. Karia and Yogita M. Parmar

Dedicated to Professor Karl-Joachim Wirths
on the occasion of his 70th birthday

Abstract. The paper is first in the sequel of investigation of Hilbert pro-C^*-modules. The main objective of the current paper is to develop the tools for operators on a locally Hilbert space. We remove the bottleneck to define the compact operators on a locally Hilbert space by avoiding the spatial intervention of mapping bounded sequences to the ones having convergent subsequence. This is at the cost of losing the simplicity of $K(H)$. We also define the classes of algebras of operators, like trace class, Hilbert Schmidt operators, Schatten class operators in general, and compact operators. All of these are complete lmc *-algebras. In passing we develop the polar decomposition of an operator on a locally Hilbert space.

Keywords. Locally Hilbert space, locally m-convex H^*-algebra, pro-C^*-algebra.

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1. Introduction and preliminaries

A *topological algebra* is a topological vector space A, which also has a ring multiplication compatible with the vector space operations such that the ring multiplication is jointly continuous. A topological algebra with a continuous involution $x \in A \mapsto x^* \in A$ is a *topological *-algebra*. A *locally m-convex *-algebra* (*lmc *-algebra*) is a topological *-algebra, the topology of which is generated by a separating family $\Gamma(A)$ of submultiplicative *-seminorms. An lmc-algebra A is called proper if for $x \in A$, $Ax = 0$ implies $x = 0$. For $p \in \Gamma(A)$, $N_p = \{x \in A : p(x) = 0\}$ is a closed *-ideal of A. A_p denotes the completion of $(A/N_p, |||\cdot|||_p)$, where $|||x + N_p|||_p = p(x)$, $x \in A$. $\pi_p : A \to A_p$, $\pi_p(x) = x + N_p$ is a continuous *-homomorphism. We denote $x+N_p$ by x_p. A seminorm p on A is a *C^*-seminorm* if $p(x^*x) = p(x)^2$. A *pro-C^*-algebra* is a complete lmc *-algebra, the topology of which is generated by a family of C^*-seminorms. In this case, the family of all continuous C^*-seminorms is denoted by $S(A)$ and for each $p \in S(A)$,

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$A_p = A/N_p$ [9]. For $p \leq q$, $\pi_{pq} : A_q \to A_p$, defined by $\pi_{pq}(x_q) = x_p$ is a surjective C^*-morphism. From [9], $A = \varprojlim_{p \in S(A)} A_p$. A pro-$C^*$-algebra is said to be *simple* if $\{0\}$ and A are its only closed ideals. An element x of a pro-C^*-algebra A is called *positive* if $x = x^*$ and $sp(x) \subset [0, \infty)$, where $sp(x)$ denotes the spectrum of x. Also, we denote $|x| = (x^*x)^{1/2}$. In the set up of non-normed topological *-algebras, inverse limits of C^*-algebras have been extensively studied by Phillips [9], Fragoulopoulou [4], and Bhatt and Karia [1], [2], [3] and Karia [7] under various names like inverse limits of C^*-algebras, locally C^*-algebras and pro-C^*-algebras. Basically the objects of a pro-C^*-algebra are unbounded. A *locally m-convex H^*-algebra* is a locally m-convex algebra whose topology is generated by a family $\Gamma(A)$ of submultiplicative seminorms each of which is induced by a pseudo inner product, i.e., for all $p \in \Gamma(A)$, there exists a pseudo inner product $(\cdot, \cdot)_p$ such that $p(x)^2 = (x, x)_p$ and for any $x \in A$ there is an $x^* \in A$ such that $(xy, z)_p = (y, x^*z)_p$, $(yx, z)_p = (y, zx^*)_p$ for every $y, z \in A$ and $p \in \Gamma(A)$. The element x^* (not necessarily unique) is called an *adjoint* of x. If A is proper, then x^* is unique and the correspondence $x \mapsto x^*$ (on A) defines an involution on A [5, Theorem 1.3]. For a complete locally m-convex H^*-algebra A, A_p, $p \in \Gamma(A)$, is a proper Banach algebra, having therefore a continuous involution. In this case $A = \varprojlim_{p \in \Gamma(A)} A_p$ [5, Theorem 2.3]. Locally m-convex H^*-algebras are investigated by Haralampidou [5].

In the case of pro-C^*-algebra A, the representation of A into bounded linear operators is disadvantageous losing a lot of information of elements of such an algebra due to their inherent unboundedness. For this reason, [3] develops the representation theory of pro-C^*-algebras into unbounded operators. Realizing a pro-C^*-algebra A as an inverse limit of C^*-algebras, representation theory of pro-C^*-algebra A has been studied by Inoue [6], embedding A into a closed *-subalgebra $L(H)$ of all operators on a locally Hilbert space H. Throughout this note H will denote a locally Hilbert space and $\mathcal{H}$ will denote a classical Hilbert space and $BL(\mathcal{H})$ will denote the classical operator algebra of all bounded linear operators on $\mathcal{H}$. While investigating the Hilbert pro-C^*-modules in our further program, we need to develop certain concepts in $L(H)$. We do it in this paper. We investigate $L(H)$ in Section 2 and decompose the operators in $L(H)$, using Arens-Michael decomposition, making them more friendly with the inverse limit. In doing so we also repair a gap found in Arens-Michael decomposition of $L(H)$ in [4, Remarks(i), pp. 107–108]. In literature the Hilbert C^*-modules over $K(\mathcal{H})$, the C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$, have found many applications. Many results from the theory of Hilbert C^*-modules over $K(\mathcal{H})$ have been proved using the theory of Hilbert modules over the H^*-algebra $\mathscr{C}^2(\mathcal{H})$, the algebra of Hilbert Schmidt operators on a Hilbert space $\mathcal{H}$. We extend the idea of compact operators in Section 3, that also includes the atomic part of compact operators viz. the rank one operators and minimal projections in the set up of locally Hilbert spaces. In Section 4 we obtain the polar decomposition of an operator on a locally Hilbert space by exploiting the inverse limit system. Finally, in Section 5 we introduce the Schatten class operators and recapture the classical inclusion of finite rank operators sitting in Schatten class-1, trace class operators, which in turns sits in Schatten class-2, the Hilbert Schmidt operators. Continuing this program, in the sequel we intend to investigate Hilbert pro-C^*-modules over $K(H)$ and $\mathscr{C}^2(H)$, orthogonality preserving mappings on a Hilbert pro-C^*-modules, Wigner's theorem for Hilbert pro-C^*-modules.

2. The Algebra $L(H)$

We denote the norm on the Hilbert space $\mathcal{H}$ by $\|\cdot\|_{\mathcal{H}}$ and the operator norm on $BL(\mathcal{H})$ by $\|\cdot\|$. Let Λ be a directed set and for each $\alpha \in \Lambda$, let $\mathcal{H}_\alpha$ be a Hilbert space with inner product $(\cdot,\cdot)_\alpha$. We assume that the family $\{\mathcal{H}_\alpha\}_{\alpha\in\Lambda}$ of Hilbert spaces satisfies $\mathcal{H}_\alpha \subset \mathcal{H}_\beta$ and $(\cdot,\cdot)_\alpha = (\cdot,\cdot)_\beta$ on $\mathcal{H}_\alpha$ whenever $\alpha \leq \beta$. Fix $H = \cup_{\alpha\in\Lambda}\mathcal{H}_\alpha$, a vector space with the following topology.

Definition 2.1. [6] $X \subset H$ is *closed* in H if either $X = H$ or there exists $\alpha \in \Lambda$ such that X is closed in $\mathcal{H}_\alpha$. H, topologized in this way, is called a *locally Hilbert space.*

It is known [6, Lemma 5.1] that the topology described above is making H a T_1 topological space. H is nothing but the inductive limit of $\{\mathcal{H}_\alpha\}_{\alpha\in\Lambda}$.

For a linear operator $T : H \to H$, $T_{|\mathcal{H}_\alpha}$ is denoted by T_α for every $\alpha \in \Lambda$. For $\alpha \leq \beta$, $\mathcal{H}_\alpha$, being a closed subspace of $\mathcal{H}_\beta$, gives rise to an orthogonal projection $P_{\alpha\beta}$ from $\mathcal{H}_\beta$ onto $\mathcal{H}_\alpha$. In what follows by an operator on H, we mean a continuous linear $T : H \to H$ satisfying $P_{\alpha\beta}T_\beta = T_\beta P_{\alpha\beta}$ whenever $\alpha \leq \beta$. The class of such operators is denoted by $L(H)$; that is, $L(H) = \{T : H \to H : T$ is continuous, linear and $P_{\alpha\beta}T_\beta = T_\beta P_{\alpha\beta}$, whenever $\alpha \leq \beta\}$. The point here is that $\mathcal{H}_\alpha$ as well as $\mathcal{H}_\alpha^\perp$ in $\mathcal{H}_\beta$ are invariant under T_β whenever $\alpha \leq \beta$. Consequently, $T \in L(H)$ if and only if

(1) $T_\alpha \in BL(\mathcal{H}_\alpha)$ for each $\alpha \in \Lambda$,
(2) $\mathcal{H}_\alpha$ reduces T_β for every $\alpha \leq \beta$; i.e., $\mathcal{H}_\alpha$ and $\mathcal{H}_\alpha^\perp$ (in $\mathcal{H}_\beta$) are invariant under T_β.

The following is proved in [6, §5]; we provide the undocumented part of the proof, this includes mainly the completeness of $L(H)$.

Theorem 2.2. *$L(H)$ is a pro-C^*-algebra with the topology generated by the family of seminorms $p_\alpha(T) = \|T_\alpha\|_\alpha$, where $\|\cdot\|_\alpha$ denotes the operator norm on $BL(\mathcal{H}_\alpha)$.*

Proof. Let $T \in L(H)$. Since $T_\alpha \in BL(\mathcal{H}_\alpha)$, $T_\alpha^* \in BL(\mathcal{H}_\alpha)$ for every $\alpha \in \Lambda$. Since $P_{\alpha\beta}T_\beta = T_\beta P_{\alpha\beta}$ for $\alpha \leq \beta$ and $P_{\alpha\beta}$ is a projection, we get $T_\beta^* P_{\alpha\beta} = P_{\alpha\beta}T_\beta^*$ for $\alpha \leq \beta$. Consequently, $T_\beta^*(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ for every $\alpha \leq \beta$. For ξ and $\eta \in \mathcal{H}_\alpha$ we have,

$$(T_\beta^*(\xi), \eta)_\beta = (\xi, T_\beta(\eta))_\beta = (\xi, T_\alpha(\eta))_\alpha = (T_\alpha^*(\xi), \eta)_\alpha.$$

Therefore, $T_\beta^*(\xi) = T_\alpha^*(\xi)$ for each $\xi \in \mathcal{H}_\alpha$. This defines the involution $T \in L(H) \mapsto T^* \in L(H)$. Since $\|\cdot\|_\alpha$ is a C^*-norm on $BL(\mathcal{H}_\alpha)$, we conclude that p_α is a C^*-seminorm on $L(H)$ for each $\alpha \in \Lambda$.

To prove the completeness of $L(H)$, let $(T_i)_{i\in I}$ be a Cauchy net in $L(H)$ and $\alpha \in \Lambda$. Then $(T_{i\alpha} = T_{i|\mathcal{H}_\alpha})_{i\in I}$ is a Cauchy net in $BL(\mathcal{H}_\alpha)$. Since $BL(\mathcal{H}_\alpha)$ is complete, there is $T_{(\alpha)} \in BL(\mathcal{H}_\alpha)$ such that $\|T_{i\alpha} - T_{(\alpha)}\|_\alpha \to 0$. Thus, for each $\alpha \in \Lambda$ we have $T_{(\alpha)} \in BL(\mathcal{H}_\alpha)$. Now, let $\alpha, \beta \in \Lambda$ be such that $\alpha \leq \beta$ and $\xi \in \mathcal{H}_\alpha$. Then

$$P_{\alpha\beta}T_{(\beta)}(\xi) = \lim_i P_{\alpha\beta}T_{i\beta}(\xi) = \lim_i T_{i\beta}P_{\alpha\beta}(\xi) = T_{(\beta)}P_{\alpha\beta}(\xi). \tag{2.1}$$

So, this defines $T \in L(H)$ such that $T_\alpha = T_{(\alpha)}$ and $p_\alpha(T_i - T) \to 0$ for every $\alpha \in \Lambda$. Thus, $T_i \to T$. Therefore $L(H)$ is complete. ■

We record here that $T \in L(H) \mapsto T_\alpha$ is a *-homomorphism for every $\alpha \in \Lambda$. By [9] $L(H)_\alpha = L(H)/N_\alpha$ is complete for every $\alpha \in \Lambda$, where $N_\alpha = \ker p_\alpha$. Consequently, $L(H) = \varprojlim_{\alpha\in\Lambda}(L(H)_\alpha, |||\cdot|||_\alpha)$, where $|||T + N_\alpha|||_\alpha = p_\alpha(T)$.

Proposition 2.3. *$(L(H)_\alpha, |||\cdot|||_\alpha)$ is isometrically *-isomorphic to a closed *-subalgebra of $BL(\mathcal{H}_\alpha)$ for every $\alpha \in \Lambda$.*

Proof. Clearly, for each $\alpha \in \Lambda$, the map $\theta_\alpha : L(H)_\alpha \to BL(\mathcal{H}_\alpha)$, defined by $\theta_\alpha(T + N_\alpha) = T_{|\mathcal{H}_\alpha}$, $(T \in L(H))$ is a *-isomorphism satisfying $\|\theta_\alpha(T + N_\alpha)\|_\alpha = \|T_\alpha\|_\alpha = p_\alpha(T) = |||T + N_\alpha|||_\alpha$ for every $\alpha \in \Lambda$. ■

Regarding $L(H)_\alpha$ as a closed subalgebra of $BL(\mathcal{H}_\alpha)$, now onwards we shall denote $T + N_\alpha$ as T_α itself and $|||\cdot|||_\alpha$ as $\|\cdot\|_\alpha$.

In [4, Remark (i), pp 107-108], it is proved that $L(H) = \varprojlim BL(\mathcal{H}_\alpha)$ up to a topological *-isomorphism. The result has, in fact, a gap. An inadvertent error in the proof claims that the above map θ_α is onto for each $\alpha \in \Lambda$. The following provides a counter example to this showing that the map θ_α is not surjective in general.

Example 2.4. Let $\mathcal{H}_1 = \mathbb{C} \times \{0\} \times \{0\}$, $\mathcal{H}_2 = \mathbb{C}^2 \times \{0\}$ and $\mathcal{H}_3 = \mathbb{C}^3$. Then $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3$ and $H = \cup_{i=1}^3 \mathcal{H}_i$ is a locally Hilbert space. Clearly $H = \mathbb{C}^3$ as a vector space. We first describe $L(H)$. Let $(x, y, z) \in \mathbb{C}^3$. Then $(x, y, z) =$

$x(1,0,0) + y(0,1,0) + z(0,0,1)$. Fix $T \in L(H)$. Assume that $T(1,0,0) = (a,b,c), T(0,1,0) = (d,e,f)$ and $T(0,0,1) = (g,h,i)$. Then $T(x,y,z) = (ax + dy + gz, bx + ey + hz, cx + fy + iz)$. Now,

$$\begin{aligned}(ax + dy, bx + ey, cx + fy) &= T_3 P_{23}(x,y,z) \\ &= P_{23} T_3(x,y,z) \\ &= (ax + dy + gz, bx + ey + hz, 0).\end{aligned}$$

Hence, $gz = 0, hz = 0$ and $cx + fy = 0$, giving $g = h = 0$. Similarly, from $P_{12}T_2 = T_2P_{12}$, we get $d = b = 0$. Therefore, $T(x,y,z) = (ax, ey, iz)$. Thus,

$$L(H) = \{T : \mathbb{C}^3 \to \mathbb{C}^3 : T(x,y,z) = (ax, by, cz); a, b, c \in \mathbb{C}\}.$$

Let $S \in BL(\mathcal{H}_2)$ be defined by $S(x,y,0) = (y,x,0)$. If $\theta_2 : L(H)_2 \to BL(\mathcal{H}_2)$ is onto, then there exists $T \in L(H)$ such that $T_2 = \theta_2(T + N_2) = S$ and $T_{2|\mathcal{H}_1} = S_{|\mathcal{H}_1} = T_1$. But $T_2(x,0,0) = (0,x,0) \notin \mathcal{H}_1$, a contradiction. Thus, θ_2 cannot be onto.

Let $\alpha \in \Lambda$. Set $\mathcal{L}_\alpha = \overline{\text{span}}(\cup_{\beta<\alpha}\mathcal{H}_\beta)$ and $\mathcal{K}_\alpha = (\mathcal{L}_\alpha)^\perp$ in $\mathcal{H}_\alpha$. Clearly, $\mathcal{H}_\alpha = \mathcal{L}_\alpha \oplus \mathcal{K}_\alpha$. Also notice that for any $\gamma < \alpha$, $\mathcal{K}_\gamma$ is a closed subspace of $\mathcal{L}_\alpha$. Therefore, $\oplus_{\gamma<\alpha}\mathcal{K}_\gamma = \mathcal{M}_\alpha$ (say) is also a closed subspace of $\mathcal{L}_\alpha$. Thus, $\mathcal{L}_\alpha = \mathcal{M}_\alpha \oplus \mathcal{J}_\alpha$, where $\mathcal{J}_\alpha = \mathcal{M}_\alpha^\perp$ in $\mathcal{L}_\alpha$. The following is straightforward

Lemma 2.5. *For any $\alpha \in \Lambda$, $\mathcal{H}_\alpha = \mathcal{M}_\alpha \oplus \mathcal{J}_\alpha \oplus \mathcal{K}_\alpha$.*

Theorem 2.6. *Let $T \in L(H)$. Then for any $\alpha \in \Lambda$, $T_\alpha = T_{|\mathcal{H}_\alpha} = T_\alpha^{\mathcal{M}} \oplus T_\alpha^{\mathcal{J}} \oplus T_\alpha^{\mathcal{K}}$, where $T_\alpha^{\mathcal{M}} = T_{|\mathcal{M}_\alpha}$, $T_\alpha^{\mathcal{J}} = T_{|\mathcal{J}_\alpha}$ and $T_\alpha^{\mathcal{K}} = T_{|\mathcal{K}_\alpha}$.*

Proof. First we show that T keeps $\mathcal{K}_\alpha$ invariant for every $\alpha \in \Lambda$. Indeed let $\alpha \in \Lambda$ and $\xi \in \mathcal{H}_\alpha$. Then $\xi = \xi_\alpha^{\mathcal{M}} \oplus \xi_\alpha^{\mathcal{K}} \oplus \xi_\alpha^{\mathcal{J}}$, where $\xi_\alpha^{\mathcal{M}} \in \mathcal{M}_\alpha, \xi_\alpha^{\mathcal{K}}$ and $\xi_\alpha^{\mathcal{J}} \in \mathcal{J}_\alpha$. Since T is continuous and for any $\beta < \alpha$, $\mathcal{H}_\beta$ is a reducing subspace of $\mathcal{H}_\alpha$ with respect to T_α, we have $T_\alpha(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha$, $T_\alpha(\mathcal{K}_\alpha) \subset \mathcal{K}_\alpha$. Consequently, $T(\mathcal{J}_\alpha) \subset \mathcal{J}_\alpha$. Thus,

$$\begin{aligned}T_\alpha(\xi) &= T_\alpha(\xi_\alpha^{\mathcal{M}} \oplus \xi_\alpha^{\mathcal{J}} \oplus \xi_\alpha^{\mathcal{K}}) \\ &= T_\alpha(\xi_\alpha^{\mathcal{M}}) \oplus T_\alpha(\xi_\alpha^{\mathcal{J}}) \oplus T_\alpha(\xi_\alpha^{\mathcal{K}}) \\ &= (T_\alpha^{\mathcal{M}} \oplus T_\alpha^{\mathcal{J}} \oplus T_\alpha^{\mathcal{K}})(\xi).\end{aligned}$$

This completes the proof. ∎

Remark 2.7. In view of [6, Theorem 5.1] we can assume that $\mathcal{J}_\alpha = \{0\}$ for all $\alpha \in \Lambda$. Consequently, we have the following.

Corollary 2.8. *For any $\alpha \in \Lambda$, $\mathcal{H}_\alpha = \mathcal{M}_\alpha \oplus \mathcal{K}_\alpha = \oplus_{\gamma\leq\alpha}\mathcal{K}_\gamma$.*

Summarizing all the above, we have the following.

Theorem 2.9. *Let $T \in L(H)$. Then $T = \oplus_{\gamma\in\Lambda} T_\gamma^{\mathcal{K}}$ and for any $\alpha \in \Lambda$, $T_\alpha = T_\alpha^{\mathcal{M}} \oplus T_\alpha^{\mathcal{K}} = \oplus_{\gamma\leq\alpha} T_\gamma^{\mathcal{K}}$, where $T_\alpha^{\mathcal{M}} = T_{|\mathcal{M}_\alpha}$ and $T_\alpha^{\mathcal{K}} = T_{|\mathcal{K}_\alpha}$.*

3. The Algebra $K(H)$

For a Hilbert space $\mathcal{H}$, $F(\mathcal{H})$ denotes the set of all finite rank operators on $\mathcal{H}$, and $K(\mathcal{H})$ denotes the set of all compact operators on $\mathcal{H}$. It is known that $F(\mathcal{H})$ is dense in $K(\mathcal{H})$. We use this to define compact operators on a locally Hilbert space H. We denote the class of finite rank operators that is, the operators having finite dimensional range, on H by $F(H)$. It is evident that $F(H)$ is a *-ideal of $L(H)$. Suppose that $E \in L(H)$ be a rank one operator. By Theorem 2.9, $E = \oplus_{\gamma\in\Lambda} E_\gamma^{\mathcal{K}}$ with exactly one $E_\beta^{\mathcal{K}} \neq 0$. Since $R(E_\beta^{\mathcal{K}}) \subset R(E)$, there exist $\xi_\beta, \eta_\beta \in \mathcal{K}_\beta$ such that $E_\beta^{\mathcal{K}} = \xi_\beta \otimes \eta_\beta$. Thus, $p_\alpha(E) = \|\xi_\beta\|_{\mathcal{H}_\alpha} \|\eta_\beta\|_{\mathcal{H}_\alpha}$ if $\beta \leq \alpha$, otherwise zero. Conversely, it is easy to see that given an $\alpha \in \Lambda$, and $\xi_\alpha, \eta_\alpha \in \mathcal{K}_\alpha$, the linear map $\xi_\alpha \otimes \eta_\alpha : H \to H$ defined by

$$\xi_\alpha \otimes \eta_\alpha(\zeta) = (\zeta_\alpha, \eta_\alpha)_\alpha \xi_\alpha, (\zeta \in H)$$

is of rank one and continuous, commuting with all $P_{\beta\gamma}$, for all β, $\gamma \in \Lambda$ with $\beta \leq \gamma$. Here ζ_α is the component of ζ in $\mathcal{K}_\alpha$.

The following is apparent.

Lemma 3.1. *Let $\xi_\alpha, \xi'_\alpha, \eta_\alpha, \eta'_\alpha \in \mathcal{K}_\alpha$ and $T \in L(H)$. Then the following hold.*

(1) $(\xi_\alpha \otimes \xi'_\alpha)(\eta_\alpha \otimes \eta'_\alpha) = (\eta_\alpha, \xi'_\alpha)_\alpha (\xi_\alpha \otimes \eta'_\alpha)$.
(2) $(\xi_\alpha \otimes \eta_\alpha)^* = \eta_\alpha \otimes \xi_\alpha$.
(3) $T(\xi_\alpha \otimes \eta_\alpha) = T(\xi_\alpha) \otimes \eta_\alpha$.
(4) $(\xi_\alpha \otimes \eta_\alpha)T = \xi_\alpha \otimes T^*(\eta_\alpha)$.

Remark 3.2. For a unit vector $\xi_\alpha \in \mathcal{K}_\alpha$, $\xi_\alpha \otimes \xi_\alpha$ is a rank one projection. Conversely, for each rank one projection E, there exists $\alpha \in \Lambda$ and a unit vector $\xi_\alpha \in \mathcal{K}_\alpha$ such that $E = \xi_\alpha \otimes \xi_\alpha$.

Theorem 3.3. *$F(H)$ is linearly spanned by the rank one projections.*

Proof. If $T \in F(H)$, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $T = \oplus_{i\leq n} T_{\alpha_i}^{\mathcal{K}}$, where $T_{\alpha_i}^{\mathcal{K}} \in BL(\mathcal{K}_{\alpha_i})$. Since $R(T_{\alpha_i}^{\mathcal{K}}) \subset R(T)$ for all $i \leq n$, $T_{\alpha_i}^{\mathcal{K}}$ is a finite rank operator on Hilbert space $\mathcal{K}_{\alpha_i}$ for all $i \leq n$. By [8, Theorem 2.4.6] $T_{\alpha_i}^{\mathcal{K}}$ is linearly spanned by rank one projections on $\mathcal{K}_{\alpha_i}$ for all $i \leq n$ completing the proof. ■

We define $K(H)$, the set of compact operators on a locally Hilbert space, to be $\overline{F(H)}$. Since addition, scalar multiplication, multiplication and involution are continuous in a pro-C^*-algebra, $K(H)$ being a closure of an ideal, is a closed *-ideal in $L(H)$. Let $\alpha \in \Lambda$ and $K(H)_\alpha = K(H)/(N_\alpha \cap K(H))$. Then it is easy to see that $K(H) = \varprojlim K(H)_\alpha$ and that $K(H)_\alpha \subset K(\mathcal{H}_\alpha)$ via $\theta_{\alpha|K(H)_\alpha}$, for every $\alpha \in \Lambda$. We know that the C^*-algebra of all compact operators on a Hilbert space is simple, but in case of locally Hilbert space it is not true. The example is provided below.

Example 3.4. Let H be as in Example 2.4. Then clearly $K(H) = L(H)$. Consider $I = \{T \in L(H) : T(x, y, z) = (ax, by, 0), a, b \in \mathbb{C}\}$. Then I is a closed *-ideal of $L(H)$. It is also proper as rank one projection $(0, 0, 1) \otimes (0, 0, 1) \notin I$. Therefore, $K(H)$ is not simple.

Along the line of C^*-algebra following [8], we define a minimal projection in a pro-C^*-algebra.

Definition 3.5. Let A be a pro-C^*-algebra. A nonzero $e \in A$ is said to be a projection if $e = e^* = e^2$. Further, it is called a minimal projection if $eAe = \mathbb{C}e$.

Theorem 3.6. *Let A be a pro-C^*-algebra. A nonzero $e \in A$ is a minimal projection if and only if for every $p \in S(A)$ either $e_p = 0$ or e_p is a minimal projection in A_p.*

Proof. Let $e \in A$ be a minimal projection, $p \in S(A)$ and $e \notin N_p$. Since π_p is a *-homomorphism from A onto A_p, we have $e_p = \pi_p(e^2) = (\pi_p(e))^2 = {e_p}^2$ and $e_p^* = (\pi_p(e))^* = \pi_p(e^*) = e_p$. Also for $a_p \in A_p$, $e_p a_p e_p = \pi_p(eae) = \pi_p(\lambda e) = \lambda \pi_p(e) = \lambda e_p$, which gives $e_p A_p e_p = \mathbb{C}e_p$. Thus, e_p is a minimal projection in A_p. Conversely, let $e \in A$ be such that for each $p \in S(A)$ either $e_p = 0$ or a minimal projection in A_p. Let $a \in A$ and $p, q \in S(A)$, with $p < q$ and $e_p \neq 0$. Suppose $e_p a_p e_p = \lambda_p e_p$ and $e_q a_q e_q = \lambda_q e_q$. Then $\lambda_p e_p = e_p a_p e_p = \pi_{pq}(e_q a_q e_q) = \pi_{pq}(\lambda_q e_q) = \lambda_q e_p$. Thus, $\lambda_p = \lambda_q = \lambda$(say). We observe that if p and q are not comparable and $e_p \neq 0 \neq e_q$, then we find $s \geq p, q$ to assert that $\lambda_p = \lambda_s = \lambda_q$. Therefore, we get λ in $\mathbb{C}$ such that $eae = \lambda e$. Thus, $eAe = \mathbb{C}e$. This completes the proof. ∎

Remark 3.7. It is known that minimal projections in $BL(\mathcal{H})$ are rank one projections. From the above Theorem and Theorem 2.9 it is clear that $E \in L(H)$ is a minimal projection if and only if there exist $\alpha \in \Lambda$, and a unit vector $\xi_\alpha \in \mathcal{K}_\alpha$ such that $E = \xi_\alpha \otimes \xi_\alpha$.

Theorem 3.8. *Let $S \in L(H)$ be a selfadjoint operator. Then $S = 0$ if and only if $ESE = 0$ for all minimal projections of $L(H)$.*

Proof. Fix $\alpha \in \Lambda$ and ξ_α, a unit vector of $\mathcal{K}_\alpha$. From Lemma 3.1 and the above remark we have

$$0 = (\xi_\alpha \otimes \xi_\alpha) S (\xi_\alpha \otimes \xi_\alpha) = (\xi_\alpha \otimes \xi_\alpha)(S(\xi_\alpha) \otimes \xi_\alpha) = (S_\alpha^{\mathcal{K}}(\xi_\alpha), \xi_\alpha)\xi_\alpha \otimes \xi_\alpha.$$

Therefore, $(S^{\mathcal{K}}_\alpha(\xi_\alpha), \xi_\alpha) = 0$. Since $\alpha \in \Lambda$ and $\xi_\alpha \in \mathcal{K}_\alpha$ are arbitrary, we have $(S^{\mathcal{K}}_\alpha(\xi_\alpha), \xi_\alpha) = 0$ for all $\alpha \in \Lambda$ and $\xi_\alpha \in \mathcal{K}_\alpha$. Since $S^{\mathcal{K}}_\alpha$ is self adjoint, $S^{\mathcal{K}}_\alpha = 0$ for all $\alpha \in \Lambda$ and hence $S = 0$. ∎

4. Polar Decomposition in $L(H)$

Before we obtain the polar decomposition of an operator $T \in L(H)$, we need to define isometry on a locally Hilbert space. $T \in L(H)$ is called an *isometry* on a locally Hilbert space H if T_α is an isometry on $\mathcal{H}_\alpha$ for every $\alpha \in \Lambda$. The following defines the partial isometry on a locally Hilbert space analogous to [8, p-50].

Definition 4.1. Let H be a locally Hilbert space. $T \in L(H)$ is said to be a *partial isometry* if for each $\alpha \in \Lambda$, $T_\alpha \in BL(\mathcal{H}_\alpha)$ is a partial isometry.

Theorem 4.2. *Let $T \in L(H)$. Then there exists unique partial isometry $S \in L(H)$ such that $T = S|T|$ and $\ker S_\alpha = \ker T_\alpha$ for every $\alpha \in \Lambda$. Moreover $S^*T = |T|$.*

Proof. Let $T \in L(H)$. Fix $\alpha \in \Lambda$. Define $S_0^\alpha : |T_\alpha|(\mathcal{H}_\alpha) \to \mathcal{H}_\alpha$ by $S_0^\alpha(|T_\alpha|(\xi)) = T_\alpha(\xi)$. It is clear S_0^α is linear. Also,

$$\begin{aligned} \||T_\alpha|(\xi)\|^2_{\mathcal{H}_\alpha} &= (|T_\alpha|(\xi), |T_\alpha|(\xi))_\alpha \\ &= (|T_\alpha|^2(\xi), \xi)_\alpha \\ &= (T_\alpha^* T_\alpha(\xi), \xi)_\alpha \\ &= (T_\alpha(\xi), T_\alpha(\xi))_\alpha \\ &= \|T_\alpha(\xi)\|^2_{\mathcal{H}_\alpha}. \end{aligned}$$

Therefore, S_0^α is well defined and is an isometry. Thus, it has a unique isometric linear extension (also denoted by S_0^α) to $\overline{|T_\alpha|(\mathcal{H}_\alpha)}$. Let $T_\alpha = S^\alpha|T_\alpha|$ be the polar decomposition of T_α in $BL(\mathcal{H}_\alpha)$, where

$S^\alpha = \begin{cases} S_0^\alpha & \text{on } \overline{|T_\alpha|(\mathcal{H}_\alpha)}, \\ 0 & \text{on } \overline{|T_\alpha|(\mathcal{H}_\alpha)}^\perp. \end{cases}$ It is apparent that $\ker S^\alpha = \ker T_\alpha$ and $S^{\alpha *}T_\alpha = |T_\alpha|$. Thus, we get S^α satisfying the above for every $\alpha \in \Lambda$. Now we show that $S^\beta_{|\mathcal{H}_\alpha} = S^\alpha$ and $P_{\alpha\beta}S^\beta = S^\beta P_{\alpha\beta}$ whenever $\alpha \leq \beta$. If $\xi \in \mathcal{H}_\alpha \cap \overline{|T_\beta|(\mathcal{H}_\beta)}^\perp$, then $0 = S^\beta(\xi) = T_\beta(\xi) = T_\alpha(\xi)$. Since $\ker S^\alpha = \ker T_\alpha$, we have $S_\alpha(\xi) = 0$. Now suppose $\xi \in \overline{|T_\beta|(\mathcal{H}_\beta)} \cap \mathcal{H}_\alpha$. Then for some $\eta \in \mathcal{H}_\beta$, $S^\beta(\xi) = S_0^\beta(|T_\beta|(\eta)) = T_\beta(\eta) = T_\alpha(\eta) = S_0^\beta(|T_\alpha|(\eta)) = S^\alpha(\xi)$. Therefore $S^\beta_{|\mathcal{H}_\alpha} = S^\alpha$. To show that $P_{\alpha\beta}S^\beta = S^\beta P_{\alpha\beta}$, it is enough to show that $P_{\alpha\beta}S^\beta = S^\beta P_{\alpha\beta}$ on $|T_\beta|(\mathcal{H}_\beta)$. Let

$\xi = |T_\beta|(\eta)$ for some $\eta \in \mathcal{H}_\beta$. Then $\eta = \eta_\alpha + \eta'$, where $\eta_\alpha \in \mathcal{H}_\alpha$ and $\eta' \in \mathcal{H}_\alpha^\perp$ in $\mathcal{H}_\beta$. Now for any $\alpha \leq \beta$,

$$\begin{aligned} P_{\alpha\beta}S^\beta(\xi) &= P_{\alpha\beta}(S_0^\beta(|T_\beta|(\eta_\alpha + \eta'))) \\ &= P_{\alpha\beta}[S_0^\beta(|T_\beta|(\eta_\alpha)) + S_0^\beta(|T_\beta|(\eta'))] \\ &= P_{\alpha\beta}(T_\beta(\eta_\alpha) + T_\beta(\eta')) \\ &= T_\beta(\eta_\alpha) \\ &= T_\alpha(\eta_\alpha) \\ &= S_0^\alpha(|T_\alpha|(\eta_\alpha)) \\ &= S_0^\beta(|T_\alpha|(\eta_\alpha)) \\ &= S_0^\beta(|T_\beta|(\eta_\alpha)) \\ &= S^\beta P_{\alpha\beta}(\xi) \end{aligned}$$

Therefore, $S^\beta P_{\alpha\beta} = P_{\alpha\beta}S^\beta$ whenever $\alpha \leq \beta$. Hence, there is a unique $S \in L(H)$ such that $S_\alpha = S_{|\mathcal{H}_\alpha} = S^\alpha$. Thus, S is a partial isometry. Since $T_\alpha = S_\alpha|T_\alpha|$ and $S_\alpha^* T_\alpha = |T_\alpha|$ for every $\alpha \in \Lambda$, we have $T = S|T|$ and $\ker T_\alpha = \ker S_\alpha$ for every $\alpha \in \Lambda$. This completes the proof. ∎

5. Schatten Class Operators

Let $T \in L(H)$. Let $\alpha \in \Lambda$ and E_α be an orthonormal basis of $\mathcal{H}_\alpha$. Then we define the α^{th} Hilbert Schmidt operator seminorm $p_{2,\alpha}(T)$ as

$$p_{2,\alpha}(T) = \|T_\alpha\|_{2,\alpha} = \left(\sum_{\xi \in E_\alpha} \|T_\alpha(\xi)\|_{\mathcal{H}_\alpha}^2 \right)^{1/2},$$

where $\|T_\alpha\|_{2,\alpha}$ is the Hilbert-Schmidt norm of $T_\alpha \in BL(\mathcal{H}_\alpha)$. Since Hilbert-Schmidt norm of T_α is independent of choice of orthonormal basis of $\mathcal{H}_\alpha$ [8], the definition of $p_{2,\alpha}(T)$ is also independent of choice of orthonormal basis of $\mathcal{H}_\alpha$. An operator $T \in L(H)$ is said to be a *Hilbert-Schmidt operator* if $p_{2,\alpha}(T) < \infty$ for every $\alpha \in \Lambda$. We denote the class of such operators as $\mathscr{C}^2(H)$.

It is apparent that $T \in \mathscr{C}^2(H)$ implies $T_\alpha \in \mathscr{C}^2(\mathcal{H}_\alpha)$ for every $\alpha \in \Lambda$. Passing onto $\|\cdot\|_{2,\alpha}$ on $BL(\mathcal{H}_\alpha)$, the following is easily verified.

Proposition 5.1. *Let $T, S \in \mathscr{C}^2(H)$ and $\lambda \in \mathbb{C}$. Then for every $\alpha \in \Lambda$, following hold.*

(1) $p_{2,\alpha}(T + S) \leq p_{2,\alpha}(T) + p_{2,\alpha}(S)$ *and* $p_{2,\alpha}(\lambda T) = |\lambda| p_{2,\alpha}(T)$.
(2) $p_{2,\alpha}(T) = p_{2,\alpha}(T^*)$.
(3) $p_\alpha(T) \leq p_{2,\alpha}(T)$.

(4) $p_{2,\alpha}(TS) \leq p_\alpha(T)p_{2,\alpha}(S)$ *and* $p_{2,\alpha}(TS) \leq p_{2,\alpha}(T)p_\alpha(S)$.

The following theorem puts $\mathscr{C}^2(H)$ in a proper perspective among non-normed topological algebras.

Theorem 5.2. *$\mathscr{C}^2(H)$ is a complete locally m-convex H^*-algebra with respect to the topology generated by the family of Hilbert-Schmidt seminorms $\{p_{2,\alpha}\}_{\alpha\in\Lambda}$.*

Proof. From Proposition 5.1, it is clear that $\{p_{2,\alpha}\}$ is a separating family of submultiplicative *-seminorm on $\mathscr{C}^2(H)$. Consequently, in view of definition of $\mathscr{C}^2(H)$, $\mathscr{C}^2(H)$ is a locally m-convex *-algebra with respect to the topology generated by $\{p_{2,\alpha}\}_{\alpha\in\Lambda}$. Defining $(T,S)_{2,\alpha} = tr(T^*_\alpha S_\alpha)$, $T, S \in \mathscr{C}^2(H)$, $\alpha \in \Lambda$, where $tr(\cdot)$ denotes the trace of an operator, it follows that $(\cdot,\cdot)_{2,\alpha}$ defines a pseudo inner product on $\mathscr{C}^2(H)$ such that $p_{2,\alpha}(T)^2 = (T,T)_{2,\alpha}$, $(TS,P)_{2,\alpha} = (S,T^*P)_{2,\alpha}$ and $(ST,P)_{2,\alpha} = (S,PT^*)_{2,\alpha}$. To prove completeness of $\mathscr{C}^2(H)$, let $(T_i)_{i\in I}$ be a Cauchy net in $\mathscr{C}^2(H)$ and $\alpha \in \Lambda$. Then $(T_{i\alpha} = T_{i|\mathcal{H}_\alpha})$ is a Cauchy net in $\mathscr{C}^2(\mathcal{H}_\alpha)$. Since $\mathscr{C}^2(\mathcal{H}_\alpha)$ is complete, there exists $T_{(\alpha)} \in \mathscr{C}^2(\mathcal{H}_\alpha)$ such that $\|T_{i\alpha} - T_{(\alpha)}\|_{2,\alpha} \to 0$. Also, $\|T_{i\alpha} - T_{(\alpha)}\| \leq \|T_{i\alpha} - T_{(\alpha)}\|_{2,\alpha}$ shows that $(T_{i\alpha})_{i\in I}$ is a net in $L(H)_\alpha$ which converges to $T_{(\alpha)} \in BL(\mathcal{H}_\alpha)$. Since $L(H)_\alpha$ is complete, $T_{(\alpha)} \in L(H)_\alpha$. Thus for each $\alpha \in \Lambda$ we have $T_{(\alpha)} \in L(H)_\alpha$. Thus there exists unique $T \in L(H)$ such that $T_\alpha = T_{(\alpha)}$. Now, $p_{2,\alpha}(T) = \|T_\alpha\|_{2,\alpha} = \|T_{(\alpha)}\|_{2,\alpha} < \infty$ for every $\alpha \in \Lambda$ showing that $T \in \mathscr{C}^2(H)$. This completes the proof. ■

Let $N_{2,\alpha} = \{T \in \mathscr{C}^2(H) : p_{2,\alpha}(T) = 0\}$. Then it can be readily seen that $N_{2,\alpha}$ is closed *-ideal in $\mathscr{C}^2(H)$. Also, $\mathscr{C}^2(H)/N_{2,\alpha} \hookrightarrow \mathscr{C}^2(\mathcal{H}_\alpha)$ via $\theta_{2,\alpha}$, where $\theta_{2,\alpha}(T + N_{2,\alpha}) = T_\alpha$. The map $\theta_{2,\alpha}$ is a well-defined *-homomorphism because $N_{2,\alpha} \subset N_\alpha$. Denote the closure of $\theta_{2,\alpha}(\mathscr{C}^2(H)/N_{2,\alpha})$ in $\mathscr{C}^2(\mathcal{H}_\alpha)$ by $\mathscr{C}^2(H)_\alpha$. From Arens-Michael decomposition we have $\mathscr{C}^2(H) = \varprojlim_{\alpha\in\Lambda}(\mathscr{C}^2(H)_\alpha, \|\cdot\|_{2,\alpha})$. Our next aim is to show that $\mathscr{C}^2(H) \subset K(H)$ for a locally Hilbert space H.

Lemma 5.3. *$F(H) \subset \mathscr{C}^2(H)$, where $F(H)$ is the set of all finite rank operators of $L(H)$.*

Proof. It is enough to show that $\mathscr{C}^2(H)$ contains every rank one operator of $L(H)$. Let $T = \xi_\beta \otimes \eta_\beta$, $\xi_\beta, \eta_\beta \in \mathcal{K}_\beta$, $\beta \in \Lambda$ be any rank one operator. Then

$$p_{2,\alpha}(T) = \left(\sum_{\zeta\in E_\alpha} \|\xi_\beta \otimes \eta_\beta(\zeta)\|^2_{\mathcal{H}_\alpha}\right)^{1/2} = \left(\sum_{\xi\in E_\alpha} \|(\zeta_\beta,\eta_\beta)_\beta \xi_\beta\|^2_{\mathcal{H}_\alpha}\right)^{1/2}, \tag{5.1}$$

where E_α is an orthonormal basis of $\mathcal{H}_\alpha$, $\alpha \in \Lambda$. Therefore,

$$p_{2,\alpha}(T) = \begin{cases} \|\xi_\beta\|_{\mathcal{H}_\beta}\|\eta_\beta\|_{\mathcal{H}_\beta}, & \text{if } \beta \leq \alpha \\ 0 & \text{otherwise .} \end{cases}$$

Thus, $p_{2,\alpha}(T) < \infty$ for every $\alpha \in \Lambda$. ■

Remark 5.4. $F(H)$ is a *-ideal of $\mathscr{C}^2(H)$.

Theorem 5.5. *$F(H)$ is dense in $\mathscr{C}^2(H)$ with respect to the topology generated by the family of $\{p_{2,\alpha}\}_{\alpha\in\Lambda}$.*

Proof. Let $V = V(p_{2,\alpha_1}, \epsilon_1) \cap V(p_{2,\alpha_2}, \epsilon_2) \cap \cdots \cap V(p_{2,\alpha_n}, \epsilon_n)$ be a basic neighbourhood in $\mathscr{C}^2(H)$, where $V(p_{2,\alpha_i}, \epsilon_i) = \{U \in \mathscr{C}^2(H) : p_{2,\alpha_i}(U) < \epsilon_i\}$, $1 \le i \le n$, and $T \in \mathscr{C}^2(H)$. Let $\epsilon' = \min_{1\le i\le n}\{\epsilon_i\}$ and $\epsilon = \epsilon'/2$. Let $\gamma \in \Lambda$ be such that $\alpha_i \le \gamma$, $1 \le i \le n$. For $\beta \le \gamma$, let G_β be an orthonormal basis of $\mathcal{K}_\beta$. Since $\mathcal{H}_\gamma = \oplus_{\beta\le\gamma}\mathcal{K}_\beta$, $\cup_{\beta\le\gamma} G_\beta = E_\gamma$ (say) is an orthonormal basis of $\mathcal{H}_\gamma$. Since $T_\gamma \in \mathscr{C}^2(\mathcal{H}_\gamma)$, we have $\left(\sum_{\zeta\in E_\gamma} \|T_\gamma(\zeta)\|^2_{\mathcal{H}_\gamma}\right)^{1/2} < \infty$. Therefore, there exists a finite set $F_\gamma \subset E_\gamma$ such that $\sum_{\zeta\in E_\gamma\setminus F_\gamma} \|T_\gamma(\zeta)\|^2_{\mathcal{H}_\gamma} < \epsilon^2$. Define $S = S^{\mathcal{K}}_{\alpha_1} \oplus S^{\mathcal{K}}_{\alpha_2} \oplus \cdots \oplus S^{\mathcal{K}}_{\alpha_n}$, where $S^{\mathcal{K}}_{\alpha_i} = T^{\mathcal{K}}_{\gamma\ |span(F_\gamma\cap\mathcal{K}_{\alpha_i})}$, $1 \le i \le n$. Then $S^{\mathcal{K}}_{\alpha_i} \in F(\mathcal{K}_{\alpha_i})$, $1 \le i \le n$. Consequently, $S \in F(H)$. Now for i, $1 \le i \le n$, observe that, $p_{2,\alpha_i}(S-T)^2 \le p_{2,\gamma}(S-T)^2 = \sum_{\zeta\in E_\gamma\setminus F_\gamma} \|T_\gamma(\zeta)\|^2_{\mathcal{H}_\gamma} < \epsilon^2$. Thus, $p_{2,\alpha_i}(S-T) < \epsilon < \epsilon_i$, $1 \le i \le n$, showing that $S - T \in V$. Therefore $S \in (T + V) \cap F(H)$. This completes the proof. ■

Remark 5.6. If $T \in \mathscr{C}^2(H)$, then there exists a net $(T_i)_{i\in I}$ of finite rank operators such that, for every $\alpha \in \Lambda$, $p_{2,\alpha}(T_i - T) \to 0$. By Proposition 5.1 we have $p_\alpha(T_i - T) \to 0$ for every $\alpha \in \Lambda$. Thus $T \in \overline{F(H)}$ and hence compact. We have $\mathscr{C}^2(H) \subset K(H)$.

For $T \in L(H)$ and $\alpha \in \Lambda$, we define $p_{1,\alpha}(T)$ as $p_{1,\alpha}(T) = (p_{2,\alpha}(|T_\alpha|^{1/2}))^2$. If E_α is an orthonormal basis of $\mathcal{H}_\alpha$, then

$$\begin{aligned} p_{1,\alpha}(T) &= \sum_{\zeta\in E_\alpha} \||T_\alpha|^{1/2}(\zeta)\|^2_{\mathcal{H}_\alpha} \\ &= \sum_{\zeta\in E_\alpha} (|T_\alpha|^{1/2}(\zeta), |T_\alpha|^{1/2}(\zeta)_\alpha \\ &= \sum_{\zeta\in E_\alpha} (|T_\alpha|(\zeta), \zeta)_\alpha. \end{aligned}$$

If $p_{1,\alpha}(T) < \infty$ for every $\alpha \in \Lambda$, then we call T a *trace class operator.* The class of such operators is denoted by $\mathscr{C}^1(H)$. As a corollary to the following is a known result.

Lemma 5.7. [8, Lemma 2.4.12] *Let $\mathcal{H}$ be a Hilbert space, $U, V \in \mathscr{C}^2(\mathcal{H})$, E be an orthonormal basis of $\mathcal{H}$ and $S = U^*V$. Then the family $((S(x), x))_{x\in E}$ is absolutely summable. That is,*

$$\sum_{x\in E} |(S(x), x)| < \infty \quad \text{and} \quad \sum_{x\in E} (S(x), x) = \frac{1}{4}\sum_{k=0}^{3} i^k \|V + i^k U\|_2^2.$$

In fact, we have the following analogous result.

Corollary 5.8. *Let $U, V \in \mathscr{C}^2(H)$ and $S = U^*V$. For every $\alpha \in \Lambda$, let E_α be an orthonormal basis of $\mathcal{H}_\alpha$. Then the family $((S_\alpha(\xi), \xi))_{\xi\in E_\alpha}$ is absolutely summable for every $\alpha \in \Lambda$. That is, for every $\alpha \in \Lambda$ $\sum\limits_{\xi\in E_\alpha} |(S_\alpha(\xi), \xi)_\alpha| < \infty$ and*

$$\sum_{\xi\in E_\alpha} (S_\alpha(\xi), \xi)_\alpha = \frac{1}{4}\sum_{k=0}^{3} p_{2,\alpha}(V_\alpha + i^k U_\alpha)^2.$$

Proof. If $U, V \in \mathscr{C}^2(H)$, then $U_\alpha, V_\alpha \in \mathscr{C}^2(\mathcal{H}_\alpha)$ for every $\alpha \in \Lambda$. Thus, by Lemma 5.7 proof follows. ■

The connection between trace class operators and Hilbert-Schmidt operators is given in the following result.

Theorem 5.9. *Let $S \in L(H)$. Then the following conditions are equivalent.*

(1) *S is a trace class operator.*
(2) *$|S|$ is a trace class operator.*
(3) *$|S|^{1/2}$ is a Hilbert-Schmidt operator.*
(4) *There exist Hilbert-Schmidt operators U, V on H such that $S = UV$.*

Proof. (1) ⇒ (2). $p_{1,\alpha}(|S|) = p_{1,\alpha}(S) < \infty$ for every $\alpha \in \Lambda$.
(2) ⇒ (3). Suppose that $|S|$ is trace class. Let E_α be any orthonormal basis of H_α for every $\alpha \in \Lambda$.

$$\begin{aligned} p_{2,\alpha}(|S_\alpha|^{1/2}) &= \sum_{\zeta\in E_\alpha} \||S_\alpha|^{1/2}(\zeta)\|^2_{\mathcal{H}_\alpha} \\ &= \sum_{\zeta\in E_\alpha} (|S_\alpha|^{1/2}(\zeta), |S_\alpha|^{1/2}(\zeta))_\alpha \\ &= \sum_{\zeta\in E_\alpha} (|S_\alpha|(\zeta), \zeta)_\alpha \\ &= p_{1,\alpha}(|S_\alpha|) < \infty. \end{aligned}$$

Thus, $|S|^{1/2}$ is a Hilbert-Schmidt operator.
(3) ⇒ (4). Let $S = T|S|$ be polar decomposition of S, where T is a partial

isometry. Then $S = T|S|^{1/2}|S|^{1/2}$. Since $\mathscr{C}^2(H)$ is an ideal of $L(H)$, $T|S|^{1/2}$ is a Hilbert-Schmidt operator. Let $U = T|S|^{1/2}$ and $V = |S|^{1/2}$. Then U and V both are Hilbert-Schmidt operators and $S = UV$.
(4) $\Rightarrow$ (1). Suppose $S = UV$ where $U, V \in \mathscr{C}^2(H)$. If $S = T|S|$ be polar decomposition of S, then $|S| = T^*S = T^*(UV) = (T^*U)V$. Let E_α be any orthonormal basis of $\mathcal{H}_\alpha$ for every $\alpha \in \Lambda$. Then by Corollary 5.8 $\sum\limits_{\zeta \in E_\alpha} (|S_\alpha|(\zeta), \zeta)_\alpha < \infty$ for every $\alpha \in \Lambda$. Thus $p_{1,\alpha}(S) < \infty$ for every $\alpha \in \Lambda$. ■

Proposition 5.10. *Let $T, S \in L(H)$ and $\lambda \in \mathbb{C}$. Then the following holds:*

(1) $p_{1,\alpha}(T + S) \leq p_{1,\alpha}(T) + p_{1,\alpha}(S)$ *and* $p_{1,\alpha}(\lambda T) = |\lambda| p_{1,\alpha}(T)$.
(2) $p_{1,\alpha}(T) = p_{1,\alpha}(T^*)$.
(3) $p_\alpha(T) \leq p_{1,\alpha}(T)$.
(4) $p_{1,\alpha}(TS) \leq p_\alpha(T) p_{1,\alpha}(S)$ *and* $p_{1,\alpha}(TS) \leq p_{1,\alpha}(T) p_\alpha(S)$.

Proof. Proof is straight forward verification followed by the fact that $p_{1,\alpha}(T)$ is trace class norm of T_α in $BL(\mathcal{H}_\alpha)$. ■

The following shows that the analogue of the trace class operators in our set up is a complete lmc *-algebra.

Theorem 5.11. *$\mathscr{C}^1(H)$ is complete lmc *-algebra with respect to the topology generated by the family of trace class seminorms $\{p_{1,\alpha}\}_{\alpha \in \Lambda}$.*

Proof. From Proposition 5.10, it is clear that $p_{1,\alpha}$ is a submultiplicative *-seminorm on $\mathscr{C}^1(H)$ for every $\alpha \in \Lambda$ and hence it is an lmc *-algebra with respect to the topology generated by the family $\{p_{1,\alpha}\}_{\alpha \in \Lambda}$ of trace class seminorms. To prove completeness of $\mathscr{C}^1(H)$, let $(T_i)_{i \in I}$ be a Cauchy net in $\mathscr{C}^1(H)$ and $\alpha \in \Lambda$. Then $(T_{i\alpha} = T_{i|\mathcal{H}_\alpha})$ is a Cauchy net in $\mathscr{C}^1(\mathcal{H}_\alpha)$. Since $\mathscr{C}^1(\mathcal{H}_\alpha)$ is complete, there exists $T_{(\alpha)} \in \mathscr{C}^1(\mathcal{H}_\alpha)$ such that $\|T_{i\alpha} - T_{(\alpha)}\|_{1,\alpha} \to 0$. Also, $\|T_{i\alpha} - T_{(\alpha)}\| \leq \|T_{i\alpha} - T_{(\alpha)}\|_{1,\alpha}$ shows that $(T_{i\alpha})_{i \in I}$ is a net in $L(H)_\alpha$ which converges to $T_{(\alpha)} \in BL(\mathcal{H}_\alpha)$. Since $L(H)_\alpha$ is complete, $T_{(\alpha)} \in L(H)_\alpha$. Thus for each $\alpha \in \Lambda$ we have $T_{(\alpha)} \in L(H)_\alpha$. Thus there exists unique $T \in L(H)$ such that $T_\alpha = T_{(\alpha)}$. Now, $p_{1,\alpha}(T) = \|T_\alpha\|_{1,\alpha} = \|T_{(\alpha)}\|_{1,\alpha} < \infty$ for every $\alpha \in \Lambda$ showing that $T \in \mathscr{C}^1(H)$. This completes the proof. ■

Let $N_{1,\alpha} = \{T \in \mathscr{C}^1(H) : p_{1,\alpha}(T) = 0\}$. Then it can be readily seen that $N_{1,\alpha}$ is closed *-ideal in $\mathscr{C}^1(H)$. Also, $\mathscr{C}^1(H)/N_{1,\alpha} \hookrightarrow \mathscr{C}^1(\mathcal{H}_\alpha)$ via $\theta_{1,\alpha}$, where $\theta_{1,\alpha}(T + N_{1,\alpha}) = T_\alpha$. Here $\theta_{1,\alpha}$ is well defined as $N_{1,\alpha} \subset N_\alpha$ and a *-homomorphism. Denote the closure of $\theta_{1,\alpha}(\mathscr{C}^2(H)/N_{1,\alpha})$ in $\mathscr{C}^1(\mathcal{H}_\alpha)$ by $\mathscr{C}^1(H)_\alpha$. From Arens-Michael decomposition we have $\mathscr{C}^1(H) = \varprojlim_{\alpha \in \Lambda} (\mathscr{C}^1(H)_\alpha, \|\cdot\|_{1,\alpha})$.

Proposition 5.12. *$F(H) \subset \mathscr{C}^1(H)$, where $F(H)$ is the set of all finite rank operators of $L(H)$.*

Proof. It is enough to show that $\mathscr{C}^1(H)$ contains every rank one operator of $L(H)$. Let $T = \xi_\beta \otimes \eta_\beta$, $\xi_\beta, \eta_\beta \in \mathcal{K}_\beta$, $\beta \in \Lambda$ be any rank one operator. Let E_α be an orthonormal basis for E_α, $\alpha \in \Lambda$. Then

$$p_{1,\alpha}(T) = \begin{cases} \|\xi_\beta\|_{\mathcal{H}_\beta}\|\eta_\beta\|_{\mathcal{H}_\beta}, & \text{if } \beta \leq \alpha \\ 0 & \text{otherwise .} \end{cases}$$

Thus, $p_{1,\alpha}(T) < \infty$ for every $\alpha \in \Lambda$. ∎

Remark 5.13. $F(H)$ is a *-ideal of $\mathscr{C}^1(H)$.

The proof of the following lemma is based on the arguments similar to that in the proof of Theorem 5.5.

Lemma 5.14. *$F(H)$ is dense in $\mathscr{C}^1(H)$ with respect to the topology generated by the family $\{p_{1,\alpha}\}_{\alpha\in\Lambda}$.*

6. Conclusion

The preceding discussion shows that

$$F(H) \subset \mathscr{C}^1(H) \subset \mathscr{C}^2(H) \subset K(H) \subset L(H).$$

In fact, this chain of inclusions is a non-commutative unbounded analogue of the classical case, $c_{00} \subset \ell^1 \subset \ell^2 \subset c_0 \subset \ell^\infty$ inclusion. All these algebras are in fact complete lmc *-algebras. We conclude from the Example 2.4 that $\mathscr{C}^2(H)_\alpha, \mathscr{C}^1(H)_\alpha$, $K(H)_\alpha$ properly sitting in $\mathscr{C}^2(\mathcal{H}_\alpha), \mathscr{C}^1(\mathcal{H}_\alpha)$ and $K(\mathcal{H}_\alpha)$ respectively, as $dim(H) < \infty$ which proves that in general $\mathscr{C}^2(H), \mathscr{C}^1(H), K(H)$ is not an inverse limit of $\mathscr{C}^2(\mathcal{H}_\alpha), \mathscr{C}^1(\mathcal{H}_\alpha)$ and $K(\mathcal{H}_\alpha)$, $\alpha \in \Lambda$, respectively.

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References

[1] S. J. Bhatt and D. J. Karia, Complete positivity, tensor products and C^*-nuclearity for inverse limits of C^*-algebras, *Proc. Indian Acad. Sci. Math. Sci.,* **101** (3) (1991), 149–167.

[2] S. J. Bhatt and D. J. Karia, Topological algebras with C^*-enveloping algebras, *Proc. Indian Acad. Sci. (Math. Sci.)*, **102** (3) (1992), 201–215.

[3] S. J. Bhatt and D. J. Karia, On an intrinsic characterization of pro-C^*-algebras and applications, *J. Math. Anal. Appl.*, **175** (1993), 68–80.

[4] M. Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Mathematics Studies **200**, Elsevier Science B. V., Amsterdam, 2005.

[5] M. Haralampidou, On locally convex H*-algebras, *Mathematica Japonica*, **38** (3) (1993), 451–460.

[6] A. Inoue, Locally C^*-algebras, Mem. Faculty of Sci., Kyushu Univ. (Ser. A), **25** (1971), 197–235.

[7] D. J. Karia, *Pro(jective Limits of) C^*-algebras*, Ph. D. Thesis, Sardar Patel University, Vallabh Vidyanagar, 1993.

[8] G. J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, San Diego, 1990.

[9] N. C. Phillips, Inverse limits of C^*-algebras, *J. Operator theory*, **19** (1) (1988), 159–195.

Dinesh J. Karia
ADDRESS:
Department of Mathematics
Sardar Patel University
Vallabh Vidyanagar, Gujarat 388120
India

E-MAIL: dineshjk@spuvvn.edu

Yogita M. Parmar
ADDRESS:
Department of Mathematics
Shree D.K.V. Arts and Science College
Jamnagar, Gujarat 361008
India

E-MAIL: yogisworld315@gmail.com