STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2015-16)

SUBJECT CODE : 15MT/PC/LA24

M. Sc. DEGREE EXAMINATION, APRIL 2016
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : LINEAR ALGEBRA
TIME : 3 HOURS **MAX. MARKS : 100**

Section-A
Answer ALL the questions (5x2=10)

1. How do you make an abelian group a module over the ring of integers?
2. If S and T are nilpotent linear transformations and if$ST = TS$, prove that ST is a nilpotent transformation.
3. Define the companion matrix of the polynomial $f(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{r-1} x^{r-1} + x^r$.
4. Prove that similar matrices have the same characteristic polynomial.
5. Define a unitary matrix and give an example.

Section-B
Answer any FIVE questions (5x6=30)

6. If A and B are submodules of an R-module M, prove that
(i) $A \cap B$is a submodule of M
(ii) $A + B = \{a + b \mid a \in A, b \in B\}$is a submodule of$M$
(iii) Is $A \cup B$ a submoduleof?Justify your answer.

7. If V is a vector space of dimension n and if $T \in A(V)$ has all its characteristic roots in F, prove that T satisfies a polynomial of degree n over F.

8. Suppose that T is in$A_F(V)$, has $p(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{r-1} x^{r-1} + x^r$ as the minimal polynomial over F. Suppose, further that V as $F[x]$-module is cyclic. Then prove that there is a basis of Vover F such that the matrix of T in this basis is
$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma_0 & -\gamma_1 & . & . & -\gamma_{r-1} \end{bmatrix}.$$

9. Prove that the characteristic and minimal polynomials of a linear transformation T have the same roots except for multiplicities.

10. Prove that every self adjoint operator on a finite dimensional inner product space has a nonzero characteristic vector.

11. Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let T_1 and T_2 be the linear transformation induced by T on V_1 and V_2, respectively. If the minimal polynomial of T_1 over F is $p_1(x)$ while that of T_2 is $p_2(x)$, then prove that the minimal polynomial for T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

12. Let V be a finite dimensional inner product space. If T and U are linear operators on V then prove the following.
(i) $(T+U)^* = T^* + U^*$
(ii) $(TU)^* = U^*T^*$
(iii) $(T^*)^* = T$.

Section-C

Answer any THREE questions (3x20=60)

13. Prove that any finitely generated module over a Euclidean ring is the direct sum of a finite number of cyclic submodules.

14. If $T \in A_F(V)$ is nilpotent, of index of nilpotent n_1, then prove that there is a basis of V in which the matrix of T has the form $\begin{bmatrix} M_n & 0 & \cdots & 0 \\ 0 & M_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_r} \end{bmatrix}$
where $n_1 \geq n_2 \geq \cdots \geq n_r$ and where $n_1 + n_2 + \cdots + n_r = \dim_F(V)$.

15. Prove that two elements S and T in $A_F(V)$ are similar in $A_F(V)$ if and only if they have the same elementary divisors.

16. Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Then prove that
(i) T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.
(ii) T is diagonalizable if and only if the minimal polynomial for T has the form $p(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$, where $c_1, c_2, \cdots, c_k$ are distinct elements of F.
(10+10)

17. (a) Let V and W be finite dimensional inner product spaces over the same field, having same dimension. If T is a linear transformation of V into W, then prove that the following are equivalent.
(i) T preserves inner product.
(ii) T is an inner product space isomorphism.
(iii) T carries every orthonormal basis for V onto an orthonormal basis for W.
(iv) T carries some orthonormal basis for V onto an orthonormal basis for W.
(b) Let V be a complex vector space and f be a form on V such that $f(\alpha, \alpha)$ is real for every α. Then prove that f is Hermitian. (12+8)