STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2008 – 09)

SUBJECT CODE: MT/PC/CA34

M. Sc. DEGREE EXAMINATION, NOVEMBER 2009
BRANCH I - MATHEMATICS
THIRD SEMESTER

COURSE : MAJOR – CORE
PAPER : COMPLEX ANALYSIS – PART II
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A **(5 X 8 = 40)**
ANSWER ANY FIVE QUESTIONS

1. Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, show that $M(r) \le M(r_1)^{\alpha} M(r_2)^{1-\alpha}$, where $\alpha = \dfrac{\log(r_2/r)}{\log(r_2/r_1)}$.

2. If $\sigma = \operatorname{Re}(s) > 1$ and $\{p_n\}_{n=1}^{\infty}$ is an ascending sequence of primes, prove that $\dfrac{1}{\xi((s)} = \prod_{n=1}^{\infty}\left(1 - p_n^{-s}\right)$.

3. Prove that a locally bounded family of analytic functions has a locally bounded derivative.

4. Prove that the functions $Z = F(w)$ which map $|w| < 1$ conformally onto polygons with angles $\alpha_k \pi (k = 1,2,...,n)$ are of the form $F(w) = c\int_0^{\infty}\prod_{k=1}^{n}(w - w_k)^{-\beta_k} + c'$, where $\beta_k = 1 - \alpha_k$, the w_k are points on the unit circle and c, c' are complex constants, $0 < \alpha_k < 2$, $-1 < \beta_k < 1$.

5. Prove that a continuous function $u(z)$ which satisfies the mean value property $U(z_0) = \dfrac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$ is necessarily harmonic.

6. Prove that any two bases of the same module are connected by a unimodular transformation.

7. Prove that the zeros $a_1, a_2, ..., a_n$ and poles $b_1, b_2, ..., b_n$ of an elliptic function f satisfy $a_1 +, a_2 + ... + a_n \equiv b_1 + b_2 + ... + b_n \pmod{M}$, M being the period module of f.

SECTION – B **(3 X 20 = 60)**

ANSWER ANY THREE QUESTIONS

8. a) Suppose that $u(z)$ is harmonic for $|z| < R,$ continuous for $|z| \leq R,$ prove that

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta \text{ for all } |a| < R.$$

b) For $\sigma = \text{Re}(s) > 1$, prove that $\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz$, where $(-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi < \text{Im} \ \log(-z) < \pi$.

9. State and prove Arzela-Ascoli theorem.

10. State and prove Riemann mapping theorem.

11. a) Prove that a discrete module consists either of zero alone, of the integral multiples nw of a single complex number $w \neq 0$, or of all linear combinations $n_1 w_1 + n_2 w_2$ with integral co-efficients of two numbers w_1, w_2 with non-real ratio w_2 / w_1.

b) Prove that the Weierstrass $\mathbb{P}$-function can be represented in the form

$$\mathbb{P}(z) = \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

12. a) Derive the differential equation satisfied by the Weierstrass ρ - function.

b) Derive Legendre's relation $\eta_1 w_2 - \eta_2 w_1 = 2\pi i$.