STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011-12 & thereafter)

SUBJECT CODE : 11MT/PC/MI24

M. Sc. DEGREE EXAMINATION, APRIL 2015
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : MEASURE THEORY AND INTEGRATION
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

Answer all the questions: **5×2=10**

1. Show that, for any set A, the outer measure $m^*(A) = m^*(A+x)$ where $A + x = \{y + x : y \in A\}$.
2. If φ is a measurable simple function then prove that $\int a\varphi dx = a\int \varphi dx$ where $a \geqslant 0$.
3. Define a ring and when is a ring to be called a σ- ring.
4. Define a positive set, negative set and null set with respect to the signed measure υ.
5. If $\subseteq X \times Y$, , then define x-section and y-section of E.

SECTION – B

Answer any five questions: **5×6=30**

6. Define the Lebesgue outer measure and prove: for any sequence of sets $\{E_i\}, m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.
7. Prove that every interval is measurable.
8. Show that, for $\alpha > 1$, $\int_0^1 \frac{x \sin x}{1+(nx)^\alpha}\, dx \rightarrow 0$ as $n \rightarrow \infty$.
9. If f is continuous on a finite interval $[a, b]$ then prove that i) f is integrable and ii) the function $F = \int_a^x f(t)dt$ is differentiable such that $F'(x) = f(x)$.
10. Define a measure μ on $\mathfrak{R}$, outer measure μ^* on $\mathcal{K}(\mathfrak{R})$ and prove that if $A, B \in \mathfrak{R}$ and $A \subseteq B$ then $\mu(A) \leq \mu(B)$.
11. Let υ be a signed measure on $[\![X, \mathcal{S}]\!]$. Prove that the pair A, B is a Hahn decomposition of the set X with respect to υ such that A is a positive set and B is a negative set with $X = A \cup B$, $A \cap B = \phi$.
12. Prove that the class of elementary sets consists of those sets which may be written as the union of finite number of disjoint measurable rectangles is an algebra.

SECTION – C

Answer any three questions: **3×20=60**

13. a) Prove that there exist a non-measurable set.

    b) Let c be any real number and let f and g be real-valued measurable functions defined on the same measurable set E then prove that $f + c, cf, f + g, f - g$ and fg are also measurable. (10+10)

14. a) State and prove Fatou's lemma.

    b) Let f and g be non-negative measurable functions, then prove that $\int f dx + \int g dx = \int (f + g) dx$. (12+8)

15. a) Define the space $L^p(\mu)$ and the L^p norm of f. Then prove that, if a, b are constants and $f, g \in L^p(\mu)$ then $af + bg \in L^p(\mu)$.

    b) Prove that the space $L^p(\mu)$, $1 \leq p < \infty$ is complete. (8+12)

16. State and prove the Radon-Nikodym theorem.

17. a) Let $[\![X, \mathcal{S}, \mu]\!]$ and $[\![Y, \mathcal{T}, v]\!]$ be $\sigma-$finite meausre spaces. For $V \in \mathcal{S} \times \mathcal{T}$ write $\phi(x) = \nu(V_x)$ and $\psi(y) = \mu(V^y)$ for each $x \in X, y \in Y$. Then prove that ϕ is $\mathcal{S}$- measurable and ψ is $\mathcal{T}$- measurable and

$$\int_X \varphi d\mu = \int_Y \psi d\nu.$$

    b) State and prove Fubini's theorem on product measure. (12+8)

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