STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011–12 & thereafter)

SUBJECT CODE: 11MT/PC/FA44

M. Sc. DEGREE EXAMINATION, APRIL 2015
BRANCH I – MATHEMATICS
FOURTH SEMESTER

COURSE : CORE
PAPER : FUNCTIONAL ANALYSIS
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION—A (5x2=10)
ANSWER ALL THE QUESTIONS

1. State open mapping theorem.
2. Let $\{e_1, e_2, ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H then prove that $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$.
3. Prove that the adjoint operation $T \to T^*$ on $B(H)$ has the property $\|T^*\| = \|T\|$.
4. Define determinant and spectrum of an operator T on a Hilbert space.
5. Define Banach Algebra.

SECTION—B (5x6=30)
ANSWER ANY FIVE QUESTIONS

6. State and prove the closed graph theorem.
7. State and prove Schwarz inequality.
8. If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then prove that the linear subspace $M + N$ is closed.
9. If T is an operator on H for which $(Tx, x) = 0$ foe all x then prove that $T = 0$.
10. If T is an operator on H then prove that T is normal if and only if its real and imaginary parts commute.
11. Let B be a basis for H and T an operator whose matrix relative to B is $[\alpha_{ij}]$. Prove that T is nonsingular if and only if $[\alpha_{ij}]$ is nonsingular.
12. If the regular elements of a Banach Algebra A are denoted by G. Prove that the mapping $x \to x^{-1}$ of G into G is continuous and is a homeomorphism of G onto itself.

SECTION—C (3x20=60)
ANSWER ANY THREE QUESTIONS

13. State and prove Hahn- Banach theorem.

14. (i) Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Prove that the following conditions are all equivalent to one another

   (1) $\{e_i\}$ is complete

   (2) $x \perp \{e_i\} \Rightarrow x = 0$

   (3) if x is an arbitrary vector in H then $x = \sum (x, e_i) e_i$

   (4) if x is an arbitrary vector in H then $\|x\|^2 = \sum |(x, e_i)|^2$.

   (ii) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$.

15. (i) If T is an operator on H then prove that the following conditions are all equivalent to one another (i) $T^*T = I$ (ii) $(Tx, Ty) = (x, y) \forall\, x, y$ (iii) $||Tx|| = ||x|| \;\forall\, x$.

   (ii) If P is the projection on a closed linear subspace M of H, then prove that M reduces an operator T if and only if $TP = PT$.

16. State and prove the spectral theorem.

17. (i) If the regular elements of a Banach Algebra A are denoted by G. Prove that G is an open set.

   (ii) Prove that the boundary of the set of all singular elements S is a subset of the set of all topological divisors of zero Z.