STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011-12)

SUBJECT CODE : 11MT/MC/VL64

B. Sc. DEGREE EXAMINATION, APRIL 2014
BRANCH I – MATHEMATICS
SIXTH SEMESTER

COURSE : MAJOR CORE
PAPER : VECTOR SPACES AND LINEAR TRANSFORMATIONS
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

ANSWER ALL QUESTIONS. **(10 X 2 =20)**

1. Give two examples of vector spaces over the field of complex numbers.
2. List out all subspaces of $\mathbb{R}^2$ over $\mathbb{R}$.
3. Write down the condition for any two points in $\mathbb{R}^2$ to be a basis for $\mathbb{R}^2$ over $\mathbb{R}$.
4. Define the dual of a vector space.
5. If $S, T \in Hom(V,W)$ and $v_i S = v_i T$ for all v_i of a basis of V, prove that $S = T$.
6. Define an inner product space and give an example.
7. Define an orthonormal set of vectors in an inner product space.
8. Show that 0 is an characteristic root of $T \in A(V)$ if and only if T is singular.
9. Define the rank of a linear transformation. If $T : V \rightarrow V$ is an isomorphism of an n - dimensional space V, then what is the rank of T?
10. Define the matrix of a linear transformation $T \in A(V)$ with respect to a basis of V.

SECTION –B

ANSWER ANY FIVE QUESTIONS. **(5x8 = 40)**

11. If $v_1, v_2, ..., v_n$ are in V then prove that either they are linearly independent or some v_k is a linear combination of $v_1, v_2, ..., v_{k-1}$.
12. If S, T are subsets of a vector space V , prove that
    (i) $S \subseteq T$ implies $L(S) \subseteq L(T)$.
    (ii) $L(L(S)) = L(S)$.
    (iii) $L(S \cup T) = L(S) + L(T)$.
13. State and prove Schwarz inequality in an inner product space.
14. If V is a finite dimensional space over F, then prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0.
15. If $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct characteristic roots of $T \in A(V)$ and if $v_1, v_2, ..., v_n$ are characteristic vectors of T belonging to $\lambda_1, \lambda_2, ..., \lambda_n$ respectively, then prove that $v_1, v_2, ..., v_n$ are linearly independent over F.

16. Proof : Let U and V be vector spaces with bases $B = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots , \boldsymbol{u_n} \}$ and $B' = \{ \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots , \boldsymbol{v_m} \}$ and $T: U \rightarrow V$ a linear transformation. If $\boldsymbol{u}$ is a vector in U with image $T(\boldsymbol{u})$, having coordinate vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ relative to these bases, then $\boldsymbol{b} = A\boldsymbol{a}$, where $A = [T\{\boldsymbol{u_1})_{B'} \ldots\ldots T(\boldsymbol{u_n})_{B'}]$.
17. Let V be a vector space over a field F and 0 be the zero element of V. Prove that (i) $\alpha 0 = 0$, for every $\alpha \in F$ (ii) $0v = 0$, for every v in V (iii). $v(-\alpha) = -(\alpha v)$, for every $\alpha \in F, v \in V$ (iv) If $v \neq 0$, then $\alpha v = 0$ implies that $\alpha = 0$.

SECTION –C

ANSWER ANY TWO QUESTIONS. **(2x20 = 40)**

18. (a) Let $V = \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in R\}$, $W_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and $W_2 = \{(0, 0, z) \mid z \in \mathbb{R}\}$. Prove that V is a vector space over $\mathbb{R}$ and W_1, W_2 are subspaces of V. Further, prove that V is a direct sum of W_1 and W_2 .
    (b) Prove that any two vector spaces of same dimension are isomorphic.
    (c) Define annihilator $A(W)$ of a subspace W of a space V and prove that $A(W)$ is a subspace of $\hat{V}$. (7+8+5)

19. (a) If V and W are vector spaces of dimensions of dimensions m and n respectively over F , prove that $Hom(V, W)$ is of dimensions mn over F.
    (b) Prove that any finite dimensional inner product space V has an orthonormal space. (10 + 10)

20. (a) If V is finite dimensional over F and if $S,\ T \in A(V)$, then prove that
    (i). $r(ST) \leq r(T)$.
    (ii). $r(TS) \leq r(T)$.
    (iii). $r(ST) = r(TS) = r(T)$ for S regular in $A(V)$.
    (b) Define Similar matrices and give examples.
    (c) Define a diagonalizable matrix. (15+3+2)