STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2009 – 10)

SUBJECT CODE: MT/PC/CA34

M. Sc. DEGREE EXAMINATION, NOVEMBER 2010
BRANCH I - MATHEMATICS
THIRD SEMESTER

COURSE : CORE
PAPER : COMPLEX ANALYSIS
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A **(5 X 8 = 40)**

ANSWER ANY FIVE QUESTIONS

1. If u_1 and u_2 are harmonic in a region Ω then prove that $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ for every cycle γ which is homologous to zero in Ω.

2. Obtain a product representation of $\sin \pi z$.

3. Prove that $\zeta(S)\Gamma(S) = \int_0^\infty \frac{x^{S-1}}{e^x - 1}\, dx$.

4. Prove that the family F of analytic functions is totally bounded if and only if to every compact set $E \subset \Omega$ and every $\varepsilon > 0$, it is possible to find $f_{1,} \dots f_n \in F$ such that evry $f \in F$ satisfies $d(f, f_j) < \varepsilon$ for some f_j.

5. Prove that the functions $z = F(w)$ which map $|w| < 1$ conformally onto polygons with angles $\alpha_k \pi (k = 1,2, \dots n)$ are of the form $F(w) = C \int_0^\infty \prod_{k=1}^n (w - w_k)^{-\beta_k}\, dw + C'$ where $\beta_k = 1 - \alpha_k,\ w_k$ are points on the unit circle and C and C' are complex constants.

6. Prove that (i) the sum of the residues of an elliptic function is zero.
   (ii) A non constant elliptic function has equally many poles as it has zeors.

7. Show that $\wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left[\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right]^2$

SECTION – B **(3 X 20 = 60)**

ANSWER ANY THREE QUESTIONS

8. a) Suppose that $u(z)$ is harmonic for $|z| < R$, continuous for $|z| \le R$, prove that

$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z) \ d\theta$ for all $|a| < R$.

b) State and prove Jensen's formula.

(10+10)

9. a) If $\sigma = Re(s) > 1$ and $\{p_n\}_{n=1}^{\infty}$ is an ascending sequence of primes, prove that

$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty}(1 - p_n^{-s})$.

b) Derive the Legendre's Duplication formula if $\xi(s) = \frac{1}{2}\ s\ (1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is entire and satisfies the relation $\xi(s) = \xi(1-s)$.

(10+10)

10. State and prove Arzela Ascoli's theorem.

11. State and prove Riemann mapping theorem.

12. a) Show that an elliptic function without poles is a constant. Prove that the zeros $a_1, \dots a_n$ and poles $b_1, \dots b_n$ of an elliptic function satisfy

$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j \ (mod\ M)$.

b) If $\wp(z)$ denotes the Weierstrass $\wp$ function show that

$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$.