STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2009 – 10 & thereafter)

SUBJECT CODE: MT/PC/CA34

M. Sc. DEGREE EXAMINATION, NOVEMBER 2011
BRANCH I - MATHEMATICS
THIRD SEMESTER

COURSE : CORE
PAPER : COMPLEX ANALYSIS
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A **(5 X 8 = 40)**

ANSWER ANY FIVE QUESTIONS

1. If u_1 and u_2 are harmonic in a region Ω, then prove that $\int_r u_1 * du_2 - u_2 * du_1 = 0$ for every cycle γ which is homologous to zero in Ω.
2. Represent $\sin \pi z$ in the form of a canonical product.
3. If $\sigma = Re(s) > 1$ and $\{p_n\}$ is an ascending sequence of primes, prove that $\frac{1}{\varsigma(s)} = \prod_{n=1}^{\infty}(1 - {p_n}^{-s})$.
4. Prove that $\varsigma(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$
5. Prove that the family $\mathcal{F}$ is totally bounded if and only if to every compact set $E \subset \Omega$ and every $\varepsilon > 0$ it is possible to find $f_1, f_2, \dots, f_n \in \mathcal{F}$ such that every $f \in \mathcal{F}$ satisfies $d(f_i, f_j) < \varepsilon$ on E for some f_j.
6. Prove that the functions $z = F(w)$ which map $|w| < 1$ conformally onto polygons with angles $\alpha_k \pi (k = 1,2, \dots n)$ are of the form $F(w) = C \int_0^\infty \prod_{k=1}^{n}(w - w_k)^{-\beta_k} dw + C'$, where $\beta_k = 1 - \alpha_k$, w_k are points on the unit circle and C, C' are complex constants.
7. Prove that a discrete module consists either of zero alone, of the integral multiples $n\omega$ of a single complex number $\omega \neq 0$, or of all linear combinations $n_1\omega_1 + n_2\omega_2$ with integral co-efficients of two numbers ω_1, ω_2 with non-real ratio $\frac{\omega_2}{\omega_1}$.

SECTION – B **(3 X 20 = 60)**

ANSWER ANY THREE QUESTIONS

8. a) Suppose that $u(z)$ is harmonic for $|z| < R$, continuous for $|z| \leq R$, prove that $u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2}\, u(z)\, d\theta$ for all $|a| < R$.
   b) State and prove Jensen's formula.

9. a) Prove that $\varsigma(x) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\varsigma(1-s)$. Using this result derive the Legendre's duplication formula.
   b) Show that the function $\xi(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(s/2)\,\varsigma(s)$ is entire satisfying $\xi(s) = \xi(1-s)$ and derive the Legendre's duplication formula. (10+10)

10. State and prove Arzela-Ascoli theorem.

11. State and prove Riemann mapping theorem.

12. a) Prove that the zeros $a_1, a_2, \dots a_n$ and the poles $b_1, b_2, \dots b_n$ of an elliptic function f satisfy $a_1 + a_2 + \cdots + a_n \equiv b_1 + b_2 + \cdots + b_n$ (mod M), M being the period module of f.
    b) Prove that $\mathcal{P}(z+u) = -\mathcal{P}(z) - \mathcal{P}(u) + \frac{1}{4}\left(\frac{\mathcal{P}'(z)-\mathcal{P}'(u)}{\mathcal{P}(z)-\mathcal{P}(u)}\right)^2$. (10+10)