STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011-12 & thereafter)

SUBJECT CODE : 11MT/PC/MI24

M. Sc. DEGREE EXAMINATION, APRIL 2014
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : MEASURE THEORY AND INTEGRATION
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

Answer all the questions: **5×2=10**

1. Show that for any set A, $m^*(A) = m^*(A + x)$ where $A + x = \{y + x, \quad y \in A\}$.
2. Show that if f is a non negative measurable function, then $f = 0$ a.e if $\int f\, dx = 0$.
3. Define (i) measure (ii) complete measure (iii) $\sigma -$ finite measure
4. Define (i) signed measure (ii) positive set with respect to signed measure (iii) a negative set.
5. If $\mathcal{Y}$ is any class of subset of X then show that there exists a smallest monotone class denoted by $\mathcal{M}_0(\mathcal{Y}\ )$ containing .

SECTION – B

Answer any five questions: **5×6=30**

6. Prove that for any sequence of sets $\{E_i\} m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.
7. State and prove Lebesgue Monotone Convergence Theorem.
8. Show that if $\alpha > 1$, $\int_0^1 \frac{x \sin x}{1+(nx)^{\alpha}}\, dx = o(n^{-1})$ as $n \to \infty$.
9. Show that $L^{\infty}(\mu)$ is complete.
10. Prove that a countable union of sets positive with respect to a signed measure v is a positive set.
11. Prove: Let μ be a signed measure on $[[X, \delta]]$ and let v be of finite valued signed measure on $[[X, \delta]]$ such that $v \leq \mu$. Then given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|v|(E) > \varepsilon$ whenever $|v|(E) < \delta$.
12. Prove that the class of elementary sets ξ is an algebra.

SECTION – C

Answer any three questions: **3×20=60**

13. **a**) Prove that the class $\mathcal{M}$ is a $\sigma-$ sigebra.

    b) Prove that every interval is measurable.

14. a) Let ϕ and ψ be simple functions which vanish outside a set of finite measure, then prove that $\int(a\phi + b\psi) = a\int\phi + b\int\psi$ and if $\phi \geq \psi$ a.e. then $\int\phi \geq \int\psi$.

    b) Let f be a non-negative measurable function. Then prove that there exists a sequence $\{\phi_n\}$ of measurable simple functions such that for each $x, \phi_n(x) \uparrow f(x)$.

15. a) If μ is a measure on a $\sigma-$ ring $\mathcal{S}$, then prove that the class $\bar{\mathcal{S}}$ of sets of the form $E\Delta N$ for any sets E, N such that $E \in \mathcal{S}$ while N is contained in some sets in $\mathcal{S}$ of zero measure is a $\sigma-$ ring, and the set function $\bar{\mu}$ defined by $\bar{\mu}(E\Delta N) = \mu(E)$ is a complete measure on $\bar{\mathcal{S}}$.

    b) Let $[[X, \mathcal{S}, \mu]]$ be a measure space and f a non-negative measurable function. Then show that $\phi(E) = \int_E f\, d\mu$ is a measure on the measurable space $[[X, \mathcal{S}]]$. Also prove: $\int f\, d\mu < \infty$ then $\forall\varepsilon > 0, \exists\delta > 0$ such that, if $A \in \mathcal{S}$ and $\mu(A) < \delta$, then $\phi(A) < \varepsilon$.

16. a) If v be a signed measure on $[[X, \mathcal{S}]]$, then prove that there exists measures v^+ and v^- on $[[X, \mathcal{S}]]$ such that $v = v^+$ - v^- and v^+ perpendicular to v^-. The measures v^+ and v^- are uniquely defined by v, and $v = v^+ - v^-$ called the Jordan decomposition of v.

    b) Let f be a non-negative function and let $\phi(x) = \int_Y f_x\, dv$, $\psi(y) = \int_X f^y\, d\mu$, for each $x \in X, y \in Y$. Then prove that ϕ is $\mathcal{S}-$ measurable ψ is $\tau-$ measurable and $\int_X \phi d\mu = \int_{X\times Y} f d(\mu \times v) = \int_Y \Psi dv$.

17. a) If v_1 and v_2 are $\sigma-$ finite measures on $[[X, \delta]]$ and $v_1 \ll \mu$, $v_2 \ll \mu$, then prove that $\frac{d(v_1+v_2)}{d\mu} = \frac{dv_1}{d\mu} + \frac{dv_2}{d\mu}$ $[\mu]$.

    b) State and prove Lebesgue Decomposition Theorem.