CHAPTER 0

I N T R O D U C T I O N

0.1. The concept of weighing designs and the origin of the weighing problem

The concept of weighing designs can be made very clear with the help of the following example: Let there be two objects with weights β_1 and β_2, respectively, which are to be estimated with the help of a chemical balance that is free from any bias. Let the variance of an individual weighing be σ^2. If the two objects are weighed together on one pan of the balance, and then, on opposite pans, the equations for estimating the unknown weights β_1 and β_2 will be

$$\beta_1 + \beta_2 = y_1$$

$$\beta_1 - \beta_2 = y_2 ,$$

where, y_1 and y_2 are the corresponding readings from the scale. Thus, the estimates of the weights are $(y_1+y_2)/2$ and $(y_1-y_2)/2$. The variances of these estimates will each be $\sigma^2/2$ which is half the value of the variance when the two objects are weighed individually. The mean of the estimates of each weight, found from n pairs of weighings of sums and differences will have variance $\sigma^2/2n$, whereas, the number of pairs of weighings of the objects individually

will have to be $2n$ to give the same precision to the average estimated weight.

Thus, it is clear that, when measurements have to be made on several quantities, greater precision in estimating the measurements is possible when the measurements are made on sets of objects rather than on individuals.

The problem of weighing a number of objects may be regarded as the problem of the estimation of the effects of a number of factors which do not interact.

The 'weighing problem', i.e., the problem of determining the weights of several objects by weighing them in suitable combinations, took its origin in the following casual illustration furnished by Yates (1935): A chemist has seven light objects to weigh, and the scale also requires a zero correction. The obvious technique is to weigh each of the seven objects separately and to make an eighth weighing with no object on the scale so that the zero correction can be determined. Thus, the weight of each object will be determined as the difference between the readings of the scale when carrying that object and when empty. If the standard error of each weighing is denoted by σ (assuming that systematic errors are non-existent and that the errors are random), then, the variance of each estimated weight will

work out to $2\sigma^2$.

Yates (1935) suggested an improvement over this customary technique. This improved technique consists of weighing all the seven objects together and also weighing them in groups of three so chosen that each object is weighed four times altogether, twice with any other object and twice without it. Denoting the unknown weights of the objects as β_1, β_2, ..., β_7, and the readings from the scale as y_1, y_2, ..., y_8, the equations for determining the unknown weights according to the scheme proposed by Yates are as follows:

$$\begin{aligned}
\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 &= y_1 \\
\beta_1 + \beta_2 + \beta_3 &= y_2 \\
\beta_1 + \beta_4 + \beta_5 &= y_3 \\
\beta_1 + \beta_6 + \beta_7 &= y_4 \\
\beta_2 + \beta_4 + \beta_6 &= y_5 \\
\beta_2 + \beta_5 + \beta_7 &= y_6 \\
\beta_3 + \beta_4 + \beta_7 &= y_7 \\
\beta_3 + \beta_5 + \beta_6 &= y_8
\end{aligned} \qquad (0.1.1)$$

The weight β_1 is determined as

$$\beta_1 = \frac{y_1 + y_2 + y_3 + y_4 - y_5 - y_6 - y_7 - y_8}{4} \qquad (0.1.2)$$

Similar expressions can be obtained for the other weights.

As the variance of a sum of independent observations is the sum of the variances, the variance of the estimated weight β_1 is, from (0.1.2), $3\sigma^2/16 = \sigma^2/2$, which is only one fourth that obtained by the direct method. Thus, the improved technique has halved the standard error of each estimated weight. To get the standard error of each estimated weight as $\sigma/\sqrt{2}$ by the direct method, one has to repeat the eight weighings, i.e., one for each object and one with no object, four times and take the average of the four estimates thus obtained. Thus, if a certain degree of accuracy is required, calling for repetition of the weighings by the direct method a certain number of times, only one fourth as many weighings will be needed by Yates' method to procure the same accuracy in the average.

A further improvement in the weighing technique for the above problem was suggested by Hotelling (1944) who modified Yates' procedure to include on the other pan of the balance those objects that are not weighed. Calling the readings from the scale now as y_1', y_2', ..., y_8', the equations similar to (0.1.1) become

$$
\begin{aligned}
\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 &= y_1' \\
\beta_1 + \beta_2 + \beta_3 - \beta_4 - \beta_5 - \beta_6 - \beta_7 &= y_2' \\
\beta_1 - \beta_2 - \beta_3 + \beta_4 + \beta_5 - \beta_6 - \beta_7 &= y_3' \\
\beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_5 + \beta_6 + \beta_7 &= y_4' \\
-\beta_1 + \beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6 - \beta_7 &= y_5' \\
-\beta_1 + \beta_2 - \beta_3 - \beta_4 + \beta_5 - \beta_6 + \beta_7 &= y_6' \\
-\beta_1 - \beta_2 + \beta_3 + \beta_4 - \beta_5 - \beta_6 + \beta_7 &= y_7' \\
-\beta_1 - \beta_2 + \beta_3 - \beta_4 + \beta_5 + \beta_6 - \beta_7 &= y_8' \qquad (0.1.3)
\end{aligned}
$$

In this case, some of the y_i' may be negative. In Yates' method, the objects are always put on one pan and the weights (known) are added to the other pan to balance the two pans. The weight added to the other pan in the ith weighing operation is y_i. In Hotelling's method, when certain objects are put on one pan, and all the others, on the other pan, the weight added to either pan to maintain balance is recorded positive only when it is added to the 'other' pan. Otherwise, it is recorded as negative. Thus, the y_i' may be positive or negative. From (0.1.3), the estimate of β_1 is obtained as

$$\beta_1 = \frac{y_1' + y_2' + y_3' + y_4' - y_5' - y_6' - y_7' - y_8'}{8}, \qquad (0.1.4)$$

with similar expressions for the other unknowns. The variance of each estimated weight by this method is $\sigma^2/8$. Thus, the

standard error of the estimate is half that obtained by Yates' method. Here, the number of repetitions required to procure a particular standard error in the mean is one sixteenth that required by the direct method.

Such improvements in the technique of measurements can also be applied to other types of measurements as of distances, lengths, voltages and resistances, concentrations of chemicals in solutions, in fact, any measurement in which the measure of a combination is a known linear function of the separate measures with numerically equal coefficients. However, for the sake of simplicity, the problem will be discussed in the language of weighing operations. Further, it is quite reasonable to assume that the recorded results of weighing operations will be independent of each other and that they will have the same standard error.

In the above example, σ was assumed to be known, whereas, in actual practice, the value of σ may be unknown and it may be necessary to find an estimate of σ from the recorded results. Getting the estimate of σ from the results of the experiment may be more satisfactory since the value of σ will then refer to the actual experiment rather than to some previous experiments which might not have been made under exactly the same conditions. But, in order to

have such an estimate of σ, it is necessary that the number of observations exceeds the number of unknowns and desirable that the excess shall be large enough to ensure a stable estimate of σ.

0.2. The statistical model for the weighing problem

The results of n weighing operations to determine the individual weights of p objects on a chemical balance with zero bias fit into the linear model

$$\underline{y} = X\underline{\beta} + \underline{e} \qquad (0.2.1)$$

where, $X = ((x_{ij}))$ $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p$, is an $n \times p$ matrix of elements $x_{ij} = +1, -1$ or 0 according as in the ith weighing operation the jth object is placed, respectively, on the left pan, right pan or none; $\underline{y}$ is the $n \times 1$ observed vector of the recorded results of the weighings; $\underline{\beta}$ is a $p \times 1$ vector representing the unknown weights of the p objects; $\underline{e}$ is an $n \times 1$ unobserved random vector such that $E(\underline{e}) = \underline{0}$ and $E(\underline{e}\underline{e}') = \sigma^2 I_n$ where E stands for Expectation, $\underline{e}'$ is the transpose of $\underline{e}$ and I_n is the $n \times n$ identity matrix. In order to estimate the unknown weights from the results of the n weighing operations it is necessary that $n \geq p$ when the balance is free from bias. If the balance has bias, then, to estimate the bias and the

unknown weights, it is necessary that $n \geq (p+1)$. The bias can always be regarded as yet another unknown weight to be estimated. Taking the bias to be the first object, the elements x_{i1} will each be unity for $i = 1, 2, \ldots, n$, i.e., the matrix X will have its first column as a column of unities when the balance has a bias that is to be estimated. X represents the design matrix and is called the weighing design. The rows of X represent the weighing operations and the columns represent the objects whose weights are to be determined. In a spring balance problem, the elements x_{ij} can assume only the values +1 or 0. Consistent with the signs that the elements x_{ij} can take, the record of the ith weighing is taken as positive or negative according as the balancing weight is placed on the right pan or the left. A spring balance design can always be regarded as a chemical balance design with no object placed on the right pan in any of the weighing operations. The spring and chemical balance weighing designs are also called one-pan and two-pan weighing designs, respectively. A chemical balance weighing design is called a strictly chemical balance weighing design if in each weighing operation neither pan is empty. In other words, for such designs X, there will be at least one $+1$ and at least one -1 in each of the n rows.

As (0.2.1) represents a linear model, all the results pertaining to the general linear model will hold good for (0.2.1). A detailed account of linear models can be obtained from Graybill (1961), Rao (1973) and Searle (1971). Some of the results that are consequences of the Gauss-Markoff model are stated below: When the design matrix X is of full rank, i.e., of rank p. $X'X$ will be non-singular and the least-squares estimates of the weights $\underline{\beta}$ are given by

$$\hat{\underline{\beta}} = (X'X)^{-1}X'\underline{y} , \qquad (0.2.2)$$

where, $(X'X)^{-1}$ stands for the inverse of the matrix $(X'X)$. The variance-covariance matrix of $\hat{\underline{\beta}}$ is given by

$$\text{Cov}(\hat{\underline{\beta}}) = (X'X)^{-1}\sigma^2. \qquad (0.2.3)$$

$(X'X)^{-1}$ is denoted by $C = ((c_{ij}))$ $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, p$. The variance of the estimated weight of the ith object is given by $c_{ii}\sigma^2$. c_{ii} which is the ith diagonal element of C is known as the <u>variance factor</u> for the ith object. If σ^2 is unknown, the unbiased estimate of σ^2 based on the least-squares estimate of $\underline{\beta}$ is given by

$$\hat{\sigma}^2 = (\underline{y}-X\hat{\underline{\beta}})'(\underline{y}-X\hat{\underline{\beta}})/(n-p) \qquad (0.2.4)$$

To derive the estimates of $\underline{\beta}$ and σ^2 as in (0.2.2) and (0.2.4), no assumption need be made about the form of the distribution of the error vector $\underline{e}$. But, some such assump-

tion will be necessary if one wants to use maximum likelihood estimation. Usually the distribution of $\underline{e}$ is taken to be normal.

Soon after the introduction of the weighing problem by Yates (1935) and Hotelling (1944), a series of articles on this subject appeared, some of the early ones being those by Kishen (1945), Rao (1945), Mood (1946) and Placket and Burman (1946) . A lot of pioneering work on the weighing problem has been done by Banerjee (1948, 1949a, 1949b, 1949c, 1949d, 1950a, 1950b, 1950c, 1951, 1952).

0.3. Optimum designs and efficiency criteria

In weighing designs, one tries to locate if possible, a design matrix X with which each c_{ii} is minimum. When this is not possible, one has to construct X to suit alternative optimality criteria.

Hotelling (1944) and subsequently Moriguti (1954) proved that the minimum possible value for each variance factor c_{ii} is $1/n$ and thus, the minimum variance for each estimated weight is σ^2/n. σ^2/n is referred to as the 'minimum minimorum' of the variance of each estimated weight. The variance of each estimated weight will attain the minimum minimorum if and only if $X'X = nI_p$. A design X for which

$X'X = nI_p$ will, therefore, be the best or optimum design. But, such a design does not exist for all values of n and p. Thus, it becomes necessary to choose an optimum weighing design for a given situation. Listed below are some efficiency criteria which will be useful in comparing different weighing designs.

Criterion 0.3.1. Of two $n \times p$ weighing designs X_1 and X_2, X_1 is superior to X_2 if the variance of each estimated weight is smaller in the case of design X_1 than in the case of design X_2, i.e., if each diagonal element of $(X_1'X_1)^{-1}$ is smaller than the corresponding diagonal element of $(X_2'X_2)^{-1}$. In some cases, it might so happen that the variances of some of the estimates are smaller when X_1 is used, whereas, the variances of the other estimates are smaller when X_2 is used. Thus, out of two designs X_1 and X_2, one may give more efficient estimates of some of the weights while the other gives more efficient estimates of the other weights.

Criterion 0.3.2. X_1 is superior to X_2 if the average of the variances of the estimated weights is smaller in the case of X_1 than in the case of X_2. This is known as the A-optimality criterion (see Kiefer, 1959). A-optimality is also known as trace-optimality since the criterion reduces to

X_1 being superior to X_2 if $tr(X_1'X_1)^{-1}$ is less than $tr(X_2'X_2)^{-1}$, $tr(X)$ standing for the trace of the matrix X. The efficiency of a given $n \times p$ design X according to the A-optimality criterion is given by Kishen (1945) as

$$\frac{\sigma^2/n}{\sigma^2 \sum_{i=1}^{p} c_{ii}/p} = \frac{p}{n \sum_{i=1}^{p} c_{ii}} \qquad (0.3.1)$$

Criterion 0.3.3. X_1 is superior to X_2 if $|X_1'X_1|$ is greater than $|X_2'X_2|$ where, $|X|$ stands for the determinant of X. This criterion is known as the D-optimality criterion (see Mood, 1946 and Kiefer, 1959). Note that maximizing $|X'X|$ amounts to minimizing $|(X'X)^{-1}|$. This criterion is based on minimizing the generalized variance of the estimated weights. According to this criterion, the efficiency of a given $n \times p$ design X is

$$\frac{|X'X|}{\max |X'X|} \qquad (0.3.2)$$

Criterion 0.3.4. X_1 is superior to X_2 if the maximum characteristic root of $(X_1'X_1)^{-1}$ is smaller than the maximum characteristic root of $(X_2'X_2)^{-1}$. Equivalently, X_1 is superior to X_2 if the minimum characteristic root of $(X_1'X_1)$ is larger than the minimum characteristic root of $(X_2'X_2)$. This criterion is known as the E-optimality cri-

terion (see Ehrenfeld, 1955 and Kiefer, 1959). This is based on the fact that if $\underline{c}'\underline{\beta}$ is a linear function of the true weights subject to the condition that $\underline{c}'\underline{c} = 1$, then, the maximum value of the variance of $\underline{c}'\underline{\hat{\beta}}$ for all choices of $\underline{c}$ satisfying $\underline{c}'\underline{c} = 1$, is $\sigma^2 \phi_{max}$, where ϕ_{max} is the maximum characteristic root of $(X'X)^{-1}$. Here, $\underline{\hat{\beta}}$ refers to the least-squares estimate of $\underline{\beta}$ obtained with the design matrix X. The efficiency of an $n \times p$ design X based on this criterion is

$$\frac{\theta_{min}}{n} \qquad (0.3.3)$$

where, θ_{min} is the minimum characteristic root of $(X'X)$.

Criterion 0.3.5. X_1 is superior to X_2 if the sum of all the elements of $(X_1'X_1)^{-1}$ is less than the sum of all the elements of $(X_2'X_2)^{-1}$. This criterion is based on the estimation of the total weight of the p objects. The variance of the estimated total weight of the p objects obtained with an $n \times p$ design X is

$$\sigma^2(\text{ sum of all the elements of } (X'X)^{-1}) \qquad (0.3.4)$$

In certain situations, other definitions of best designs may be preferred. Thus, problems may arise in which one might prefer

i) to minimize the variance factors subject to the restriction that they be equal,

ii) to minimize some function of the variance factors or

iii) to minimize only a certain subset of the diagonal elements of $(X'X)^{-1}$ on a minor of the matrix $(X'X)^{-1}$ as might be the case when one wants only rough estimates of the weights of some objects, but more accurate estimates of the others.

Gupta and Das (1977) have, for instance, introduced the following definition of an optimum design : "Of the class of all $n \times p$ weighing designs, design X is said to be <u>optimum</u> if the weight of the ith object is estimated with variance σ^2/n_i, $i = 1, 2, \ldots, p$, where n_i represents the number of times the ith object is included in the n weighing operations".

It is obvious that the efficiency of two weighing designs to measure the weights of p objects can be compared only when they both have the same number of weighing operations. Normally, one expects the efficiency of a design to improve when the number of weighing operations increases.

Banerjee (1972) observed that, in the case of chemical balance weighing designs, D- and E- optimality criteria are equivalent and that in the case of spring balance weighing designs, D-optimality implies E-optimality.

0.4. Some results on matrix theory

The following well known results on matrix algebra which are used in various Chapters of this Thesis are stated without proof :

Let A_i be a square matrix of order n_i with each of its diagonal elements as r_i and each of its off-diagonal elements as λ_i, $i = 1, 2, \ldots, m$; let F_{ij} be a matrix of dimension $n_i \times n_j$ with each of its elements as λ_{ij}, $i \neq j$, $i,j = 1, 2, \ldots, m$ and let $F_{ji} = F'_{ij}$. Let

$$\left|A_{1,\ 2,\ \ldots,\ m}\right| = \begin{vmatrix} A_1 & F_{12} & F_{13} & \cdots & F_{1m} \\ F_{21} & A_2 & F_{23} & \cdots & F_{2m} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\cdot & \cdot\cdot \\ F_{m1} & F_{m2} & F_{m3} & \cdots & A_m \end{vmatrix}$$

which is a determinant of order $N = \sum_{i=1}^{m} n_i$. Then, the value of the above determinant is given by

$$\left|A_{1,\ 2,\ \ldots,\ m}\right| = \prod_{i=1}^{m} P_i \prod_{i=1}^{m} n_i \, J \qquad (0.4.1)$$

where, $P_i = (r_i - \lambda_i)^{(n_i-1)}$, $p_i = r_i + \lambda_i(n_i-1)$, $R_i = p_i/n_i$, $i = 1, 2, \ldots, m$; $\lambda_{ij} = \lambda_{ji}$ and

$$J = \begin{vmatrix} R_1 & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1m} \\ \lambda_{21} & R_2 & \lambda_{23} & \cdots & \lambda_{2m} \\ .. & .. & .. & \cdots & .. \\ .. & .. & .. & \cdots & .. \\ \lambda_{m1} & \lambda_{m2} & \lambda_{m3} & \cdots & R_m \end{vmatrix}$$

The above determinant, which is of order m, may again admit of a similar reduction. If the weighing design X is such that $X'X = A_{1,2,\ldots m}$, there will be m sets of variance factors and, in each set, the variance factors will be equal. The n_i variance factors in the ith set will be equal to

$$(n_i-1)J_i/(r_i-\lambda_i)n_iJ \qquad (0.4.2)$$

where,

$$J_i = \begin{vmatrix} R_1 & \lambda_{12} & \lambda_{13} & \cdots\cdots & \lambda_{1m} \\ \lambda_{21} & R_2 & \lambda_{23} & \cdots\cdots & \lambda_{2m} \\ .. & .. & .. & \cdots\cdots & .. \\ .. & .. & .. & \cdots\cdots & .. \\ \lambda_{i1} & \lambda_{i2} & \lambda_{i3} & \cdots R_i' \cdots & \lambda_{im} \\ .. & .. & .. & \cdots\cdots & .. \\ \lambda_{m1} & \lambda_{m2} & \lambda_{m3} & \cdots\cdots & R_m \end{vmatrix}$$

with $R_i' = \left\{ r_i + \lambda_i(n_i-2) \right\}/(n_i-1)$.

Let $A = \begin{bmatrix} F_1 & F_2 \\ F_2' & F_3 \end{bmatrix}$ be a partitioned symmetric matrix such that A and F_1 are non-singular. Then,

$$A^{-1} = \begin{bmatrix} F_1^{-1}(I+F_2F_4^{-1}F_2'F_1^{-1}) & -F_1^{-1}F_2F_4^{-1} \\ -F_4^{-1}F_2'F_1^{-1} & F_4^{-1} \end{bmatrix} \quad (0.4.3)$$

where, $F_4 = F_3 - F_2' F_1^{-1}F_2$.

The following results can be obtained as simple cases of the above results.

If A is a matrix of the form

$$(c-d)I_p + d\, E_{p,p} \quad (0.4.4)$$

where, c and d are scalars and $E_{p,p}$ is a $p \times p$ matrix with each element as unity, then,

$$|A| = (c-d)^{(p-1)} \left\{c+(p-1)d\right\} \quad (0.4.5)$$

and $$A^{-1} = (f-g)I_p + g\, E_{p,p} \quad (0.4.6)$$

where, $$f = \left\{c+(p-2)d\right\} / (c-d) \left\{c+(p-1)d\right\} \quad (0.4.7)$$

and $$g = -d/(c-d) \left\{c+(p-1)d\right\}. \quad (0.4.8)$$

Remark If an $n \times p$ weighing design X is such that $X'X$ is of the form (0.4.4), then, obviously, X will contain $c \pm 1$'s in each of its columns, i.e., each object is weighed

(on the left or right pan) c times, and the inner product of any two columns of X will be d . Such a design X will ensure that

i) the variances of the estimated weights are equal and ii) the estimated weights are equally correlated. (0.4.9)

In fact, the variance factors will each be f given by (0.4.7) and the covariance between any two estimated weights will be g given by (0.4.8). If the estimates of the weights determined with an $n \times p$ design matrix X should satisfy (0.4.9), then, X must be of the form (0.4.4). Thus, X'X being of the form (0.4.4) is a necessary and sufficient condition for the estimates of the weights obtained with the $n \times p$ design X to satisfy (0.4.9).

0.5. Some well known block designs that can be used to construct weighing designs

i) BIB designs

A balanced incomplete block design (BIBD) is an arrangement of v elements (treatments) in b blocks each of size k ($< v$) such that

1) every element occurs at most once in a block,

2) every element occurs exactly in r blocks and

3) every pair of elements occurs together in exactly

λ blocks.

v, b, r, k, λ are called the parameters of the BIB design and they satisfy the following relations:

$$vr = bk \qquad (0.5.1)$$

$$\lambda(v-1) = r(k-1) \qquad (0.5.2)$$

$$b \geq v \qquad (0.5.3)$$

If $v = b$, then, obviously, $r = k$ and in this case the BIBD is called a symmetrical BIBD.

Given a BIBD with parameters v, b, r, k, λ, its complementary BIBD is obtained by including in each of the b blocks those $(v-k)$ treatments that are not present in the corresponding block of the original BIBD. The complementary BIBD will have parameters $v' = v$, $b' = b$, $r' = b-r$, $k' = v-k$, and $\lambda' = b-2r+\lambda$.

ii) PBIB designs

Given v elements 1, 2, ..., v, a relation satisfying the following conditions is said to be an association scheme with m classes:

1) any two elements are either 1st, 2nd .... or mth associates, the relation of association being symmetrical
2) each element has n_i ith associates, the number n_i being independent of the element chosen

3) If any two elements are ith associates, then, the number of elements that are jth associates of one and kth associates of the other, is p_{jk}^{i} and is independent of the pair of the ith associates chosen. v, n_i, i = 1, 2, ..., m, and p_{jk}^{i}, i, j, k = 1, 2, ..., m, are called the parameters of the association scheme.

Given an association scheme for v elements, a <u>partially balanced incomplete block design</u> (PBIBD) is defined as follows: If there is an association scheme with m classes with given parameters, a PBIBD with m associate classes is an arrangement of the v elements into b blocks of size k (<v) such that

1) every element occurs at most once in a block,
2) every element occurs in exactly r blocks and
3) if two elements are ith associates, then, they occur together in λ_i blocks, the number λ_i being independent of the pair of ith associates chosen.

v, b, r, k, and λ_i, i = 1, 2, ..., m, are called the parameters of the PBIB design. These parameters satisfy the following relations:

$$\text{i)}\quad vr = b\mathcal{k}\,,$$

$$\text{ii)}\quad \sum_{i=1}^{m} n_i = v - 1\,,$$

$$\text{iii)}\quad \sum_{i=1}^{m} n_i \lambda_i = r(\mathcal{k}-1)\,,$$

$$\text{iv)}\quad \sum_{k=1}^{m} p_{jk}^{i} = n_j - \delta_{ij}\,, \text{ where } \delta_{ij} = 1 \text{ if } i = j \\ = 0 \text{ otherwise,}$$

$$\text{v)}\quad n_i p_{jk}^{i} = n_j p_{ik}^{j}\,. \qquad (0.5.4)$$

In a PBIBD, b and v need not satisfy (0.5.3).

A detailed account of BIBD and PBIBD can be obtained from Raghavarao (1971).

iii) BBW designs

A balanced bipartite weighing design with parameters v, k_1, k_2, λ_1, denoted by BBWD (v,k_1,k_2,λ_1) is an arrangement of v elements into b blocks $B_i = \left\{B_i^{1}\,;\,B_i^{2}\right\}$ each with $\mathcal{k} = (k_1+k_2)$ distinct elements, the number of elements in B_i^{j} being k_j, $j = 1, 2$; $i = 1, 2, \ldots, b$, such that each element occurs in r blocks, each pair of distinct elements is linked in exactly λ_1 blocks and n-linked in exactly λ_2 blocks (see Huang, 1976). If B is a block with subsets B^1 and B^2 such that $B = \left\{B^1\,;\,B^2\right\}$, where $B^1 = \left\{a_1^{1},\right.$

$a_2^1, \ldots, a_{k_1}^1 \}$, $B^2 = \{a_1^2, a_2^2, \ldots, a_{k_2}^2 \}$, then, two elements in B are said to be linked or n-linked in B , if and only if they belong to different subsets or the same subset of B , respectively. The parameters of a BBWD satisfy the following relations :

$$\text{i)}\quad b = \lambda_1 v(v-1)/(2k_1k_2) ,$$
$$\text{ii)}\quad \lambda_2 = \lambda_1 \{k_1(k_1-1) + k_2(k_2-1)\} /(2k_1k_2) \text{ and}$$
$$\text{iii)}\quad r = \lambda_1 \ell (v-1)/(2k_1k_2) \qquad (0.5.5)$$

where, r represents the number of blocks in which each element occurs.

iv) Balanced n-ary designs

A balanced n-ary block design is an arrangement of v treatments (elements) in b blocks of size ℓ such that every treatment (element) is replicated r times and $\sum_{i=1}^{b} n_{ij} n_{im}$ is a constant, where, n_{ij} is the number of times the jth treatment occurs in the ith block, i = 1, 2, ..., b ; j = 1, 2, ..., v , and each n_{ij} can take n different positive integral values including zero. In particular, the n possible values for each n_{ij} may be taken as 0, 1, 2, ..., (n-1). A BIBD is a special case of an n-ary design with n_{ij} taking only the values 0 and 1 and $\sum_{i=1}^{b} n_{ij} n_{im}$ being

equal to λ. When n_{ij} can take the three values 0, 1, 2, the n-ary design becomes what is known as a ternary design.

For all the block designs mentioned in this Section, the incidence matrix is defined as the $b \times v$ matrix $N = ((n_{ij}))$, where, n_{ij} denotes the number of times the jth element occurs in the ith block. In the case of BIBD, PBIBD and BBWD, $n_{ij} = 1$ or 0 according as the jth element occurs or does not occur in the ith block.

0.6. Chemical balance weighing designs

In a chemical balance weighing design X, the elements x_{ij} can assume the values +1, -1 or 0. As was pointed out in Section 0.3, the $n \times p$ weighing design X for which $X'X = nI_p$ will provide the minimum variance of σ^2/n for each estimated weight. A matrix X satisfying the above condition will be optimum with respect to Criteria (0.3.1) to (0.3.4). Such a matrix can be formed by choosing any p columns of what is known as a Hadamard matrix. A Hadamard matrix H_n of order n is an nth order square matrix with elements +1 and -1 such that $H_n' H_n = n I_n$. A necessary condition for the existence of a Hadamard matrix H_n is that $n = 2$ or $n \equiv 0 \pmod 4$. n is called a Hadamard number if H_n exists. Literature on the methods of

construction of Hadamard matrices, the relation between Hadamard matrices and BIBD, the applicability of Hadamard matrices as weighing designs etc., can be found in Hadamard (1893), Paley (1933), Williamson (1944), Kishen (1945), Mood (1946), Placket and Burman (1946), Baumert, Golomb and Hall (1962) and Hall (1967). A comprehensive account of Hadamard matrices and their applications is now avilable in Hedayat and Wallis (1978).

When the number of weighing operations is not a Hadamard number, the method of effectively augmenting a Hadamard matrix with its rows was studied by Kishen (1945) and Mood (1946). A substantial contribution towards this study was made by Banerjee (1949c).

When $n = p$ and n is not a Hadamard number, Mood (1946) furnished some D-optimal designs for small values of p. The construction of optimal designs for the case $n = p$ when n is not a Hadamard number was studied by Raghavarao (1959, 1960) under the conditions given by (0.4.9). He showed that for odd n, designs X satisfying

$$X'X = (n-1)I_n + E_{n,n} \qquad (0.6.1)$$

are A- , D- and E-optimal. An nth order square matrix with elements +1 and -1 satisfying (0.6.1) was termed a P_n matrix. For the case $n \equiv 2 \pmod 4$, Raghavarao (1959,1960)

showed that designs X satisfying

$$X'X = (n-1)\,I_n \qquad (0.6.2)$$

are A- and E-optimal and that designs X satisfying

$$X'X = (n-2)\,I_n + 2\,E_{nn} \qquad (0.6.3)$$

are D-optimal. An nth order square matrix with each diagonal element as zero and each non-diagonal element as +1 or -1, satisfying (0.6.2) was termed an S_n matrix. Banerjee (1975) used the notation T_n for a square matrix of order n satisfying (0.6.3). Raghavarao (1959) proved that a necessary condition for the existence of P_n is that $n = (d^2+1)/2$ where d is an odd integer. Thus, there are many values of n for which P_n is non-existent, i.e., for n = 3, 7, 9, 11, 15 etc.. Using the concept of the Hilbert norm residue symbol and the Hasse-Minkowski invariant details of which can be found in Hall (1967), Raghavarao (1960) proved a necessary condition for the existence of S_n as $(n-1, -1)_p = 1$ for all primes p, where $(a, b)_p$ is the Hilbert norm residue symbol. Using this result it can be shown that S_n matrices are non-existent for n = 22, 34, 58, 78, etc.. It can be easily proved that a necessary condition for the existence of T_n is that $n = \left\{4 + (3f^2+4)^{\frac{1}{2}}\right\}/3$, where f is an integer. Thus, T_n exists only for a few values of n like n = 6, 66, etc..

Denoting an $n \times n$ design X satisfying the conditions given by (0.4.9) as (n,s,λ) where s denotes the number of zeros in each column and λ, the inner product of two columns, Bhaskar Rao (1966) proved the following results for odd n :

i) for $n \equiv 3 \pmod 4$ and $n > 3$, $(n,0,3)$ provides a design that is A- and E-optimum,

ii) for $n > 15$, $(n,0,3)$ is D-optimum and for $n \leq 15$, $(n,0,-1)$ is D-optimum and

iii) for $n \equiv 1 \pmod 4$ and $n > 5$, when P_n does not exist, $(n,0,5)$ is A-, D- and E-optimum.

Murty and Das (1967) discussed the use of v-ary designs in v treatments and v(v-1) blocks in constructing chemical balance weighing designs. Rao and Das (1969) provided two series of chemical balance weighing designs based on balanced ternary designs and compared their efficiencies with some existing designs. Saha and Dey (1973) also discussed the use of balanced ternary designs in the construction of chemical balance weighing designs.

Dey (1971) showed how optimum chemical balance weighing designs could be constructed by replacing 0 by -1 in the incidence matrix of a BIBD.

Gupta and Das (1977) showed how to combine two Family

(A) BIB designs, i.e., BIB designs for which $b = 4(r-\lambda)$, to get chemical balance designs with which the estimate of the weight of each object can be obtained as σ^2 divided by the number of times the object is weighed in the n weighing operations.

The use of balanced bipartite weighing designs as chemical balance designs has been discussed by me (Swamy, 1982).

0.7. Spring balance weighing designs

As pointed out in Section 0.2, the elements of a spring balance weighing design can assume only the values +1 or 0.

Mood (1946) gave a method of constructing D-optimal spring balance designs when $n = p \equiv 3 \pmod 4$ from Hadamard matrices of order $(n+1)$. The method is as follows: Let $(n+1)$ be a Hadamard number. Without loss of generality, let the elements of the first row and first column of H_{n+1} be each +1. Subtract the first row of such a H_{n+1} from each of the other rows and let the result be

$$\begin{bmatrix} E_{1,(n+1)} \\ O_{n,1} \quad K_{n,n} \end{bmatrix}$$

where, $O_{n,1}$ stands for an $n \times 1$ matrix with each element

as 0. The elements of $K_{n,n}$ will be 0 or -2. By multiplying each row of $K_{n,n}$ by $-\frac{1}{2}$ obtain a matrix L_n satisfying the relation

$$\left|H_{n+1}\right| = (-2)^n \left|L_n\right| .$$

Such an L_n, after it is ascertained that $\left|L_n\right|$ is positive (by permuting rows if necessary) provides a D-optimal spring balance design. These designs are also A-optimal with each variance factor as $4n/(n+1)^2$.

Mood (1946) also provided some D-optimal spring balance designs for small values of $n = p$, by using a method given by Williamson (1946). He also constructed D-optimal designs for $n = 5$, $p = 4$ and $n = 6$, $p = 4$.

For $n > p$, the method of getting D-optimal designs as given by Mood (1946) is contained in the following results: Let $P_{r,p}$ be a matrix whose rows are all the arrangements of r +1's and $(p-r)$ 0's, $(0 \leq r \leq p)$. $P_{r,p}$ can be denoted by P_r itself p being always evident from the context. Let Q be a matrix made up of matrices P_r arranged in vertical order with P_r used n_r times. Then, Q is a spring balance design for weighing p objects in $n = \sum_{r=0}^{p} n_r \binom{p}{r}$ weighings.

Mood proved that

i) if $p = (2k-1)$, where, k is a positive integer and

if n contains the factor $\binom{p}{k}$, then, $|Q'Q|$ will be maximized when $n_k = n/\binom{p}{k}$ and all other $n_r = 0$,

ii) if $p = 2k$ and if n contains the factor $\binom{p+1}{k+1}$, then, $|Q'Q|$ will be maximized when $n_k = n_{k+1} = n/\binom{p+1}{k+1}$ and all other $n_r = 0$.

These designs, however, require far too many weighing operations. Banerjee (1948) pointed out that spring balance designs of equivalent efficiency but requiring a smaller number of weighing operations are provided by BIBD. The incidence matrix of a BIBD with parameters v, b, r, k and λ can directly be adopted as a spring balance weighing design to weigh $p = v$ objects in $n = b$ weighings. The design X in this case will be such that $X'X$ will take the form (0.4.4) with $c = r$ and $d = \lambda$. Thus, using (0.4.6) to (0.4.8), the variance factor for each object will be

$$\left\{ r + (p-2)\lambda \right\} / (r-\lambda) \left\{ r + (p-1)\lambda \right\}$$

and the covariance factor between every pair of estimated weights will be

$$-\lambda / (r-\lambda) \left\{ r + (p-1)\lambda \right\} .$$

The advantage in using the incidence matrix of a BIBD as a weighing design is that the solutions of the equations leading to the least-squares estimates of the weights can be

reduced to a routine procedure. The estimated weights are given by (see Banerjee, 1951)

$$\hat{\beta}_i = \frac{1}{r-\lambda}\left\{ Z_i - \frac{\lambda \sum_{i=1}^{p} Z_i}{r+(v-1)\lambda} \right\}$$

$$= \frac{1}{r-\lambda}\left\{ Z_i - \frac{\lambda \sum_{i=1}^{n} y_i}{r} \right\}$$

where, Z_i, $i = 1, 2, \ldots, p$ are the elements of the vector $\mathbf{X}'\mathbf{Y}$. The L_n designs of Mood (1946) can be seen to be incidence matrices of symmetrical BIBD with parameters $v = b = n$, $r = k = (n+1)/2$ and $\lambda = (n+1)/4$.

If $(n+1)$ and $(n+5)$ are Hadamard numbers, L_n and L_{n+4} exist. These will provide suitable designs for weighing n objects in n weighings and $(n+4)$ objects in $(n+4)$ weighings. The method of constructing an $(n+m) \times n$ design for $m = 1, 2, 3$, the problem of adding same or different rows of L_n to L_n and a comparative study of the variance factors for the different situations were dealt with by Banerjee (1949b, 1950b).

Banerjee (1952) studied the use of partially balanced incomplete block designs as spring balance weighing designs. The incidence matrix of a PBIBD may be singular even when

$b \geq v$. Thus, all PBIBD can not provide non-singular designs. Banerjee (1952) pointed out that a certain efficient spring balance design provided by Mood (1946) is nothing but a PBIBD.

0.8. Repeated weighing designs

When a symmetrical BIBD is used as a spring balance weighing design to estimate the weights of $n = p$ objects, no degrees of freedom will be left for estimating the error variance. To overcome this difficulty, Banerjee (1948) suggested that the design could be repeated. Thus, the 'repeated' design will be a $2p \times p$ matrix. Even when the BIBD used is not symmetrical, 'repeating' the design can be resorted to, to get more degrees of freedom for estimating the error variance. Dey (1969) pointed out that if the parameters of the symmetrical BIBD satisfy the condition $b > 2r$, then, combining the incidence matrices of the BIBD and its complement would be better than repeating the BIBD. Kulshreshtha and Dey (1970) suggested yet another alternative to the 'repeated' design for the case when the precision of some of the estimates could be sacrificed to secure more precision for the other estimates. Banerjee (1974) suggested one more alternative - that of combining the complementary BIBD with itself - and made a comparative study

of the four different designs.

For the chemical balance problem, Dey (1972) suggested four alternatives to the 'repeated' design and compared them with respect to two particular series of BIB designs. Swamy (1981b) has suggested yet another alternative to those provided by Dey (1972).

To obtain degrees of freedom for the error variance, Das and Giri (1979) suggested the use of truncated incomplete BIB designs. Such designs are obtained by omitting from the different blocks of a BIB design a certain number, say q, of treatments. The resulting design will be a balanced design (any two treatments will continue to appear together in λ blocks) for $(v-q)$ treatments in b blocks of unequal sizes, each treatment being replicated r times.

0.9. Factorial approach to the weighing problem

Kempthorne (1948) discussed the factorial approach to the weighing problem. The concept can be illustrated with the help of the following example: Consider a 2^3 factorial experiment with the eight treatment combinations as (1), a, b, ab, c, ac, bc, abc. (A clear account of factorial designs may be obtained from Kempthorne (1952)). A half replicate of the 2^3 experiment will consist of only four

treatment combinations. If we decide to include those treatment combinations which have an even number of letters in common with ABC, then, the treatment combinations to be included are (1), ab, ac, bc. Interpreting the presence of a letter a, b or c as including the corresponding object on the left pan and, the absence of a letter as not weighing the corresponding object, we have (1) representing the weighing operation on an empty pan, ab representing the weighing operation in which the first two objects are weighed etc.. The estimated weight of each object will be half of the difference between the readings of those weighings containing that object and those, not containing it. The variance of the estimated weight of each object will be $4\,\sigma^2/4 = \sigma^2$, where, σ^2 is the variance of each weighing. If the absence of a letter is interpreted as putting the corresponding object on the right pan, then, the precision can be increased. Here, the estimated weight of each object will be one fourth of the difference between the readings of those weighings containing that object and those not containing it. The variance of each estimated weight will now be $\sigma^2/4$.

Kempthorne (1948) pointed out that although the spring balance designs L_n of Mood (1946) furnished somewhat smaller variance for the estimated weights, they had the

disadvantage that the estimates were correlated, whereas, the estimates furnished by the fractional replicates were orthogonal. However, Banerjee (1949a) observed that the designs L_n of Mood are virtually the same as the designs furnished by fractional replicates. The latter designs take account of the bias and if the weighing operation corresponding to the determination of bias is omitted (in case the balance is free from bias), the resulting design will be the same as L_n. If the design L_n is adjusted to suit estimation of bias in a biased spring balance, the estimates of the weights will turn out to be orthogonal. Thus, in the example based on half-replicate of a 2^3 design, the design matrix is seen to be

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

If $(0\ 0\ 0)$ is omitted, this becomes L_3.

Kempthorne (1948) suggested the possibility of using a three-fourth replicate as a weighing design. Following this suggestion, Banerjee (1949a, 1949b) discussed the use of fractional replicates of the form $(2^\beta - 1)/2^\beta$ $(1 \leq \beta \leq n)$ and determined the variance factors on using such a

fractional replicate.

0.10. Singular weighing designs

Due to a variety of reasons (see Raghavarao (1964)), an $n \times p$ design matrix X chosen for a certain weighing problem might turn out to be of rank less than p, thus yielding a singular weighing design. When a well known block design is used as a weighing design, the parameters of the chosen design may be such that the resulting weighing design is singular (see Banerjee (1966a), Dey (1971) and Swamy (1982)).

When the design matrix X in the model (0.2.1) is not of full rank, the normal equations $X'X\underline{\beta} = X'\underline{y}$ for estimating the weights $\underline{\beta}$ by the least-squares method will not admit of a unique solution. The normal equations have many solutions and to get any one of them, a generalized inverse G of $X'X$ can be found and the corresponding solution taken as

$$\underline{\beta}^{o} = G X'\underline{y} \quad . \qquad (0.10.1)$$

A generalized inverse (g-inverse) of a matrix A is defined as any matrix G that satisfies the equation $AGA = A$. As many generalized inverses of $X'X$ exist, the solution (0.10.1) is not unique. $\underline{\beta}^{o}$ should not be taken as an estimate of $\underline{\beta}$. It is just a solution of the normal equations.

A (linear) function of the weights $\beta_1, \beta_2, \ldots, \beta_p$ is defined as <u>estimable</u> if it is identically equal to some linear function of the expected value of the vector of observations $\underline{y}$. Many necessary and sufficient conditions for the estimability of the linear function $\underline{c}'\underline{\beta}$ where, $\underline{c}'$ is a $1 \times p$ vector, exist, one of them being that there exists an $n \times 1$ vector $\underline{\ell}$ such that

$$\underline{c}' = \underline{\ell}'X \quad . \qquad (0.10.2)$$

For obtaining the <u>best linear unbiased estimate</u> (b.l.u.e) and its variance, for an estimable function, one makes use of Gauss-Markoff theorem which states that if $\underline{c}'\underline{\beta}$ is an estimable function, its b.l.u.e. is $\underline{c}'\underline{\beta}^0$ with variance $\underline{c}'G\underline{c}\sigma^2$ where, $\underline{\beta}^0$ is any solution to the normal equations using any generalized inverse G of $X'X$. Both the estimate and its variance will be invariant to the choice of G and $\underline{\beta}^0$.

When a singular $n \times p$ weighing design X is of rank r $(r < p)$, by assuming without loss of generality that the first r columns of X are independent, Raghavarao (1964) obtained the necessary and sufficient condition for the weight of the ith object $i = 1, 2, \ldots, p$, to be estimable as $\underline{\xi}_i = 0_{(p-r),1}$, where, $\underline{\xi}_i$ is the ith column vector of order $(p-r)$ of H' with H defined as $(X_r'X_r)^{-1}X_r'X_{(p-r)}$, X_r and $X_{(p-r)}$ standing for the $n \times r$ and $n \times (p-r)$

matrices given by $X = \left[X_r \quad X_{(p-r)} \right]$. From this it follows immediately that the weight of the $(r+i)$th object, $i = 1, 2, \ldots, (p-r)$, is not estimable.

When an $n \times p$ weighing design X is singular, of rank r (say), X must be augmented by at least $(p-r)$ additional rows to get a resulting design that will be of full rank. The method of adding a row that will result in an optimum design (D-optimum) was discussed by Raghavarao (1964) for the case $r = (p-1)$. Banerjee (1966a) considered the problem when the deficiency in the rank is more than one. Hazra and Banerjee (1973) dealt with the same problem by making use of the unique Moore-Penrose g-inverse. Literature on g-inverses is available in Rao and Mitra (1971) and Searle (1971).

Comparison of two singular weighing designs will be meaningful only with respect to a function $\underline{c}'\beta$ that is estimable with both the designs. The design that provides a smaller variance for the estimate of $\underline{c}'\beta$ will naturally be considered superior.

0.11. Estimation of total weight

Banerjee (1966a) observed that a design that is optimum with respect to the estimation of the individual weights need not be so with respect to the estimation of the total weight.

To substantiate this observation he made use of the singular weighing design

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} .$$

With this design the total weight is estimable (refer Section 0.10) and the variance of the total weight is $\sigma^2/2$, whereas, if three columns of a Hadamard matrix H_4 are used, the variance of the estimated total weight works out to $3\sigma^2/4$ (variance of each estimated weight is $\sigma^2/4$ and the estimates are uncorrelated). To compare two weighing designs with respect to the estimation of the total weight one has to determine the sum of all the elements of a g-inverse of $X'X$ for each of the designs and the design for which this sum is smaller is superior (see Criterion 0.3.5 and the last paragraph of Section 0.10).

Sinha (1972) discussed at length the problem of obtaining spring balance weighing designs for estimating efficiently the total weight of a given set of objects, retaining the simultaneous estimability of all the individual weights, under the restriction that at most k objects can be weighed at a time. Dey and Gupta (1977) discussed the problem of estima-

tion of the total weight, under the same restriction, for singular spring balance weighing designs. The problem has been considered in a more general sense by Swamy (1980). A similar problem under a different restriction for chemical balance designs has been studied by Swamy (1981a).

0.12. The case when $E(\underline{e}\,\underline{e}')$ is different from $\sigma^2 I_n$

When $E(\underline{e}\,\underline{e}') = V$, where, V is an $n \times n$ non-singular matrix of known elements, not necessarily the same as $\sigma^2 I_n$, the generalized least-squares estimate of $\underline{\beta}$ is obtained as

$$\tilde{\underline{\beta}} = (X'V^{-1}X)^{-1}X'V^{-1}\underline{y} \qquad (0.12.1)$$

with

$$\text{Cov}(\tilde{\underline{\beta}}) = (X'V^{-1}X)^{-1} \qquad (0.12.2)$$

If $E(\underline{e}\,\underline{e}') = \text{diag}\,(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ where diag $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ stands for a diagonal matrix with diagonal elements as $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ respectively, and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ are unknown, estimates of these can be obtained using variance components techniques and these estimates may be used to find the generalized least-squares estimate of $\underline{\beta}$. Excellent literature on variance components estimation, Minque theory and allied topics is available in Graybill (1961), Rao (1970) and Searle (1971). The elements of the error vector $\underline{e}$ could be assumed to be comprising of two components

(to suit a special situation) and the error variances could be estimated following the variance components technique.

Banerjee (1965) studied the weighing problem under the assumption that the errors are autocorrelated, i.e., taking $E(\underline{e}\,\underline{e}') = \sigma^2 V$, where,

$$V = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \\ \rho & 1 & \rho & \dots & \rho^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \dots & 1 \end{bmatrix}$$

Raghavarao, Sodhi and Singh (1968) made use of the assumption that the variance of the errors will be proportional to the total weight on the balance, i.e., $E(\underline{e}\,\underline{e}') = c\ \mathrm{diag}(z_1, z_2, \dots, z_n)$, where, $z_i = \sum_{j=1}^{p} x_{ij}\beta_j$, and c is the constant of proportionality.

0.13. Fractional weighing designs

Consider a Hadamard matrix H_p. This would be the best chemical balance design to estimate the weights of p objects in p weighings. But, due to some reason like lack of resources, time etc., one may have just p_1 $(< p)$ weighing operations corresponding to p_1 rows of H_p. This $p_1 \times p$ design is a fraction of H_p and is a <u>fractional</u>

weighing design. With such a design which is singular, one can not obtain unique and unbiased estimates of all the individual weights. But, fractional weighing designs under randomized procedures developed by Zacks (1966) provide unbiased estimates for any linear function $\underline{c}'\underline{\beta}$ and obviously, therefore, for any weight β_i, $i = 1, 2, \ldots, p$. Some results analogous to those obtained by Zacks (1966) for randomized fractional weighing designs were obtained by Banerjee (1966) for the non-randomized fractional weighing designs.

0.14. Biased estimation in weighing designs

Sihota and Banerjee (1974) applied the biased estimation procedures provided by Hoerl and Kennard (1970) for the multiple regression model, to the weighing problem.

0.15. Recent work on weighing designs

Cheng (1980) has developed a theory of optimum designs which embraces the results of Raghavarao (1959, 1960) and Bhaskar Rao (1966) as immediate consequences and can be used to prove the optimality of some weighing designs over all possible designs with respect to a very general class of criteria. He has shown with the help of a counter-example that there is no guarantee that the best design in $\mathcal{X}^{s}_{n,p}$

is really optimal over $\mathfrak{D}_{n,p}$, where, $\mathfrak{D}_{n,p}$ is the set of all possible $n \times p$ weighing designs X and $\mathfrak{D}^{s}_{n,p}$ is the set of all $n \times p$ weighing designs X for which $X'X$ is of the form (0.4.4).

Galil and Kiefer (1980) have proved some new results on optimality of weighing designs. They proved a result characterizing optimum designs when $n \geq (2p-5)$ (for $n \geq p$). With the solutions provided by them for certain hitherto unsolved cases, the list of D-optimum designs for all $p \leq 12$ (for all $n \geq p$) is now complete.