STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2008 – 09 & thereafter)

SUBJECT CODE : MT/RC/TG105

M.Phil. DEGREE EXAMINATION, JANUARY 2011
MATHEMATICS
FIRST SEMESTER

COURSE	**:**	**CORE**	
PAPER	**:**	**TOPOLOGY AND GEOMETRY**	
TIME	**:**	**3 HOURS**	**MAX. MARKS : 100**

Answer any five questions choosing at least two questions from each section. Each question carries 20 marks:

SECTION - A

1. a) When do you say that a topological space is contractible? Show that a space X is contractible if and only if X is of the same homotopy type as a single point.
   b) Construct a connected space which is not arcwise connected.
   c) Let X be an arcwise connected space and $x_0, x_1 \in X$. Show that there exists a group isomorphism of $\pi_1(X, x_0)$ onto $\pi_1(X, x_1)$. (6+7+7)

2. a) Let $(\tilde{X}, p)$ be a covering space of X and $x \in X, \tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x$. Show that there is a natural one-to-one correspondence between $p^{-1}(\{x\})$ and the coset space $\pi_1(X, x)/p_*\pi_1(\tilde{X}, \tilde{x})$.
   b) Define deck transformation. Let $(\tilde{X}, p)$ be a regular covering space of a locally simply connected space X. Show that the group $\mathcal{G}(\tilde{X}, p)$ of deck transformations is isomorphic to the quotient group $\pi_1(X, p(\tilde{x}))/p_*(\pi_1(\tilde{X}, \tilde{x}))$. (10+10)

3. a) Let s be a K-simplex and $K^{\dagger}$ be a subdivision of s^{k-1}. Let $v \in (s)$. Show that $(v, [K^{\dagger}])$ is in general position. Furthermore show that $v * [K^{\dagger}]$ is the point set of a complex $\tilde{K}$ defined by $\tilde{K} = K^{\dagger} \cup_{s^{\dagger} \in K^{\dagger}} (s^{\dagger}, v)$ and $\tilde{K}$ is a subdivision of s.
   b) Let K be a simplicial complex. Let $K^{(1)} = \{(b(s_0), \ldots, b(s_k)) : s_0 < s_1 < \ldots < s_k ; s_0, s_1, \ldots s_k \in K\}$. Show that $K^{(1)}$ is a subdivision of K and for each $s_0, s_1, \ldots, s_r \in K$ with $s_0 < s_1 < \ldots < s_r$, $(b(s_0), \ldots, b(s_r)) \subset (s_r)$. (10+10)

4. a) Let K be a simplicial complex of dimension m. Show that $\lim_{n \to \infty} K^{(n)} = 0$.
   b) Let K be a simplicial complex. For v, a vertex of K, show that $St(v)$ is an open set in $[K]$ containing v, v is the only vertex of K which lies in $St(v)$ and $\{St(v)\}_{v \in K^o}$ is an open covering of $[K]$.
   c) Let $f : [K] \to [L]$ be continuous. Show that for $\varepsilon > 0$, there exists a subdivision K_n of K and L_m of L and a simplicial approximation $\phi : K_n \to L_m$ of f such that $d(f, \phi) < \varepsilon$. (6+4+10)

SECTION – B

5. a) Let X, Y be smooth manifolds and $\Psi : X \to Y$ be smooth maps. Define the differential $d\Psi$ of Ψ at $x \in X$. Show that $d\Psi(v) = \sum_{i=1}^{m} v(y_i \circ \Psi) \frac{\partial}{\partial y_j}$.

   b) Let X and Y be smooth and let $\Psi : X \to Y$ be a smooth map. Show that $d \circ \Psi^* = \Psi^* \circ d$.

   c) State and prove Poincare's lemma. (5+5+10)

6. Let X be a smooth manifold. Show that there exists a unique linear map $d : C^\infty(X, \mathcal{G}(X)) \to C^\infty(X, \mathcal{G}(X))$ such that

   (i) $d : C^\infty(X, \Lambda^k(X)) \to C^\infty(X, \Lambda^{k+1}(X))$

   (ii) $d(f) = df$ for $f \in C^\infty(X, \Lambda^0(X))$

   (iii) if $\mu \in C^\infty(X, \Lambda^k(X))$ and $\tau \in C^\infty(X, \mathcal{G}(X))$ then $d(\mu \wedge \tau) = (d\mu) \wedge \tau + (-1)^k \mu \wedge (d\tau)$ and

   (iv) $d^2 = 0$.

7. (a) Show that the maps $C_{l-1}(K, \mathcal{G}) \xleftarrow{\partial} C_l(K, \mathcal{G}) \xleftarrow{\partial} C_{l+1}(K, \mathcal{G})$ satisfy $\partial^2 = 0$.

   (b) If K is the 1 – skeleton of 2 – simplex, compute $H_1(K, \mathcal{I}) and\ H_1(K, \mathcal{I})$ (10+10)

8. a) Let K be a simplicial complex. Define Betti number and Euler characteristic of K. Find an expression for the Euler characteristic of K.

   b) show that $\partial^* \varphi_{\langle v_0, v_1, \ldots, v_l \rangle} = \sum_v' \varphi_{\langle v, v_0, \ldots, v_l \rangle}$ where $\sum_v'$ denotes the sum over all vertices $v \in K$ such that $(v, v_0, \ldots, v_l)$ is an $(l+1)$ simplex of K. (10+10)