STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2008–09 & thereafter)

SUBJECT CODE : MT/RC/AA105

M.Phil. DEGREE EXAMINATION, JANUARY 2011
MATHEMATICS
FIRST SEMESTER

COURSE : CORE
PAPER : ALGEBRA AND ANALYSIS
TIME : 3 HOURS **MAX. MARKS : 100**

ANSWER FOUR QUESTIONS WITHOUT OMITTING ANY SECTION:

SECTION –A

1. a) Prove that in a distributive lattice L,
   $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$
   b) Show that the normal subgroups of any group G form a modular lattice.
   c) State and prove Jordan – Hölder theorem. (5+5+15)

2. a) Define a category and give two examples.
   b) Define the following
      (i) Injection functor
      (ii) Forgetful functors
      (iii) A contravarient functor from R-modules to mod-R.
   c) Explain equivalence of categories with example. (5+10+10)

3. a) For an R-module M show that the following are equivalent.
      (i) M is artinian.
      (ii) Every quotient module of M is finitely cogenerated.
      (iii) Every nonempty set S of submodules of M has a minimal element.
   b) State and prove Wedderburn-Artin theorem. (10+15)

4. a) Define Tensor product of (i) Modules (ii) Homomorphisms
   b) Let M, N and P be R-modules over a commutative ring R. Then show that
      (i) $M \otimes_R N \simeq N \otimes_R M$ as R-modules.
      (ii) $(M \otimes_R N) \otimes_R P \simeq M \otimes_R (N \otimes_R P)$ as R-modules.

   c) Let R be a Commutative Ring with unity. Let A, B be ideals in R. Then show that
      $R/_A \otimes_R R/_B \simeq R/_{A+B}$ as R-algebras. (5+10+10)

SECTION –B

5. a) Define Lebesgue integral of a measurable function.
   b) State and prove Lebesgue's Dominated convergence theorem.
   c) If $f: X \to [0, \infty]$ is measurable, $E \in \mathfrak{M}$ and $\int_E f d\mu = 0$ then show that $f = 0$ a.e. on E. (5+15+5)

6. a) State and prove Riesz representation theorem.
   b) For a positive measure μ define $L^2(\mu), 1 \le p \le \infty$ and prove that it is a complete metric space. (10+15)

7. a) State and prove Jenson's inequality and hence deduce that $y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n} \le \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$ where $\sum \alpha_i = 1$, $\alpha_i > 0$.
   b) Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then prove the following
   (i) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$ and $A_1 \subset A_2 \subset A_3$ ....
   (ii) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$ and $A_1 \supset A_2 \supset A_3$ .... and $\mu(A_1)$ is finite. (10+15)

8. a) Let $f \in L^1$ and α, λ be real numbers then prove the following
   (i) If $g(x) = f(x)e^{i\alpha x}$, then $\hat{g}(t) = \hat{f}(t - \alpha)$
   (ii) If $g(x) = f(x - \alpha)$, then $\hat{g}(t) = \hat{f}(t)e^{-i\alpha t}$
   (iii) IF $g \in L^1$ and $h = f * g$, then $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$
   b) State and prove Plancherel theorem.

⅄⅄⅄⅄⅄⅄⅄⅄⅄