STELLA MARIS COLLEGE(AUTONOMOUS), CHENNAI-86.
(For candidates admitted in the year 2008-09)
DEPARTMENTOF MATHEMATICS
MASTER OF PHILOSOPHY
PAPER IV

MT/RO/GT2 05

GRAPH THEORY

CLASS:M.Phil
TIME:3 HOURS
MARKS: 100

ANSWER ANY FIVE QUESTIONS: (5 x 20 = 100)

1. a) Prove that a graph is bipartite if and only if it has no odd cycle.
   b) Prove that a graph G is Eulerian if it has at most one nontrivial component and its vertices all have even degree.
   c) Prove that an edge is a cut edge if and only if it belongs to no cycle. (8 + 5 + 7)

2. a) Prove that for an n-vertex graph G (with $n \geq 1$) the following are equivalent:
   i) G is connected and has no cycles.
   ii) G has no loops and has for each $u, v \in V(G)$, exactly one u-v path.
   b) Define a hypercube and explain its structure.
   c) Given a graph G and a vertex $u \in V(G)$, prove that Dijkstra's Algorithm computes $d(u, z)$ for every $z \in V(G)$. (6 + 6 + 8)

3. a) Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not a loop, then prove that $\tau(G) = \tau(G-e) + \tau(G.e)$.
   b) State and prove Matrix tree theorem. (8 + 12)

4. a) State and prove Hall's theorem.
   b) Prove that a matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.
   c) Prove that for $k > 0$, every k-regular bipartite graph has a perfect matching. (8+7+5)

5. a) State and prove Whitney's theorem.
   b) Prove that $\kappa(H_{k,n}) = k$ and hence prove that the minimum number of edges in a k-connected graph on n vertices is $\lceil kn/2 \rceil$.
   c) Prove that if G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$. (10 + 5 + 5)

6. a) State and prove Brook's theorem.
   b). Define an Independent set and clique number.
   c) Prove that the wheel W_{2n} is a 4-critical graph for each $n \geq 2$. (10 + 4 + 6)

7. a) State and prove Euler's formula and hence prove that K_5 and $K_{3,3}$ are non-planar.
   b) Prove that edges in a plane graph G form a cycle in G, if and only if the corresponding dual edges form a bond in G^*. (13 + 7)

8. a) Prove that the closure of a graph is well-defined.
   b) Prove that the wheel W_n is Hamiltonian for every $n \geq 4$.
   c) State and prove Chavatal's condition for Hamiltonian cycles. (4 + 6 + 10)