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NEAT FUZZY SUBGROUPS

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ABSTRACT

The theory of Divisible and Pure subgroups has played a major role in the field of Abelian groups [1]. Similarly, Divisible and Pure Fuzzy Subgroups are also significant and they are defined by F. I. Sidky and M. A. Mishref [6]. Neat subgroups are one of the generalizations of Pure subgroups in crisp Abelian group theory and they were introduced by K. Honda ([2], [3], [4]). Neat Fuzzy Subgroups are a generalization of Pure Fuzzy subgroups. Therefore, neat fuzzy subgroups are also having significant study. A group G is said to be divisible if $nG = G,$ for all integers n. A subgroup H of a group G is said to be neat in G if $pH = H \bigcap pG$ for all primes p, and it is Pure if the equation is satisfied by all integers. This paper makes an attempt to study of Neat Fuzzy subgroups and their properties. Some examples are also given to show their existence. Throughout this paper, all groups are additive Abelian groups. All the basic definitions and notations are referred to in [1] and [5].

Key words : Neat Subgroups, Neat Fuzzy Subgroups, Pure Fuzzy Subgroups.

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1. INTRODUCTION

This paper consists of two sections. Section-1 contains some basic definitions and theorems which are used in proving the main results of this paper. Section-2 contains the main results. It is easy to see that the closed interval [0,1] forms a complete chain lattice with respect to the operations $\leq,$ minimum and maximum. The element '0' is the greatest lower bound and 1 is the least upper bound. That is $\{[0,1], \ \leq, \vee, \wedge, 1, 0\}$ is a Complete Heyting Lattice with maximal element '1' and minimal element '0' and "$\vee$" denotes the maximum, "$\wedge$" denotes the minimum in [0,1]. A function μ from a set X into the complete lattice [0,1] is known as a fuzzy subset of X. The set of all fuzzy subsets of X is denoted by $[0,1]^X$. The set $\{\mu(x): x \in X\}$ is called the Image of μ and is denoted by $\mu(X)$. The set, $\{x : x \in X \text{ and } \mu(x) > 0\}$, is called the support of μ and is denoted by μ^*. If $\mu \in [0,1]^X$, then μ is said to have the supremum property if every subset of $\mu(X)$ has maximal element. If $\mu \in [0,1]^X$ and $a \in [0,1]$, then the $a-cut$ or $a-level\,set$ of μ is denoted by μ_a and is defined as $\mu_a = \{x \in X : \mu(x) \geq a\}$. Throughout this paper G is an additive Abelian group and e be the additive identity of G. Throughout this paper P stands for set of positive primes.

A function μ from the group G into the Lattice [0,1] is called a fuzzy subset of G and the set of all fuzzy subsets of G is denoted by I^G. If $\mu \in I^G$, then μ is called a fuzzy subgroup of G if (i) $\mu(x+y) \geq \mu(x) \wedge \mu(y), \forall x, y \in G,$ and (ii) $\mu(-x) \geq \mu(x) \forall x \in G.$ Through out this paper, the set of all fuzzy subgroups of G is denoted by $I(G)$, where $I = [0,1]$. If $\mu \in I(G)$, then we define a subset of G such that $\mu_* = \{x \in G : \mu(x) = \mu(e)\}$. It is easy to see that μ_* is a subgroup of G. If $\mu \in I^G$ and $a \in [0,1]$, then we define a subset μ_a of G as $\mu_a = \{x \in G : \mu(x) \geq a\}$ called $a-cut$ or $a-$ level set of μ. It is easy to see that μ is a fuzzy subgroup of G if and only if μ_a is a subgroup of G $\forall a \in \mu(G) \cup \{b \in [0,1] : b \leq \mu(e)\}$. If μ is a fuzzy subgroup of G, then $I(\mu)$ stands for the set of all fuzzy subgroups v of G such that $v \subseteq \mu,$ where $I = [0,1]$. That is, $I(\mu) = \{v \in I(G) : v \subseteq \mu\}$. If $\mu \in I(G)$, then $\mu^* = \{x \in G : \mu(x) > 0\}$ is a subset of G and is known as the support of μ. It is easy to see that μ^* is a subgroup of G. A fuzzy subgroup μ of G is said to have the supremum property if every subset of $\mu(G)$ has a maximal element. A fuzzy subgroup μ of a group G is called divisible if for all $x_a \subseteq \mu$ with $a > 0$, and for all positive integers n, there exists $y_a \subseteq \mu$ such that $n(y_a) = x_a$. If μ is a fuzzy subgroup of a group G, then μ is divisible if, and only if, μ_a is divisible for all a such that $0 < a \leq \mu(e)$. For $x \in G, a \in [0,1]$, we define a fuzzy subset $a_{\{x\}}$ of G which takes the value a at x and 0 elsewhere is known as a fuzzy point or fuzzy singleton and it is denoted by x_a. That is, $x_a(y) = \begin{cases} a \text{ if } y = x, \\ 0 \text{ if } y \neq x, \end{cases}$.

2. MAIN RESULTS

Definition 1: If μ and ν are fuzzy subgroups of a group G such that $\nu \subseteq \mu,$ then ν is said to be neat in μ if for all $x_a \subseteq \nu$ with $a > 0,$ for all $p \in P,$ for all $y_a \subseteq \mu,$ the equation $p(y_a) = x_a$ is solvable in ν, whenever it is solvable in μ, where P is the set of all primes. That is, $p(y_a) = x_a$ implies there exists $z_a \subseteq \nu$ such that $p(z_a) = x_a.$

Towards the characterization of neat fuzzy subgroups, we have .

Theorem 1: If μ and ν are fuzzy subgroups of a group G such that $\nu \subseteq \mu,$ then ν is neat in μ if, and only if, ν_a is neat in $\mu_a,$ for all a such that $0 < a \leq \nu(e).$

Proof : Suppose ν is neat in $\mu.$ Let $a \in (0, \nu(e)]$ and p is a positive prime number. Consider the equation $p(x) = y$ for $x \in \mu_a,$ $y \in \nu_a,$ for a positive prime p. Therefore, $x_a \subseteq \mu$ and $y_a \subseteq \nu.$ Since ν is neat in μ, the equation $p(x_a) = y_a$ implies there exists z_a in ν such that $p(z_a) = y_a.$ Hence $p(z) = y.$ Thus, the equation $p(x) = y$ is solvable in ν whenever it is solvable in μ. Hence ν_a is neat in $\mu_a.$

Conversely, suppose that ν_a is neat in μ_a for all $a \in (0, \nu(e)]$. Consider the equation $p(x_a) = y_a$ for $x_a \subseteq \mu$ and $y_a \subseteq \nu.$ Since ν_a is neat in $\mu_a,$ the equation $p(x) = y$ is solvable in ν_a whenever it is solvable in $\mu_a.$ Therefore, there exists a z in ν_a such that $p(z) = y.$ Hence, $(pz)_a = y_a$ implies $p(z_a) = y_a.$Thus, the equation $p(x_a) = y_a$ is solvable in ν whenever it is solvable in μ for all primes p. Hence ν is neat in μ .

Towards the properties of neat fuzzy subgroups we have

Lemma 2 : If μ is a fuzzy subgroup of a group G and p is a prime, then

a) $p\mu(e) = \mu(e);$ **b)** $p\mu \subseteq \mu;$ **c)** $p\mu$ is a fuzzy subgroup of G;

d) If μ has the sup property, then $p\mu(G) \subseteq \mu(G).$

Proof : **a)** We have $p\mu(e) = \vee\{\mu(y) \mid e = py\} = \vee\{\mu(y) \mid y = e\} = \mu(e).$

b) If $x \notin pG,$ then $(p\mu)(x) = 0 \leq \mu(x) \Rightarrow p\mu \subseteq \mu.$ Suppose that $x \in pG$ and $x = py$ for some $y \in G.$ Then $\mu(x) = \mu(ny) \geq \mu(y).$

Hence $(p\mu)(x) = \vee\{\mu(y) \mid y \in G, x = py\} \leq \mu(x).$

c) Case (i) : Let $x, y \in G$ be any elements such that $x, y \in pG.$ Then $x - y \in pG$ since pG is a subgroup of G. Now, consider $p\mu(x - y).$ We have,

$$\begin{aligned} p\mu(x-y) &= \vee\{\mu(z) \mid z\in G, x-y=pz\} \\ &\geq \vee\{\mu(u-v) \mid u,v\in G, x=pu, y=pv\} \\ &\geq \vee\{\mu(u)\wedge\mu(v) \mid x=pu, y=pv\} \\ &= \left(\vee\{\mu(u) \mid x=pu\}\right)\wedge\left(\vee\{\mu(v) \mid y=pv\}\right) \\ &= (p\mu)(x)\wedge(p\mu)(y). \end{aligned}$$

Therefore, $p\mu$ is a fuzzy subgroup of G.

Case (ii) : Suppose $x\notin pG$ or $y\notin pG$. Then $(p\mu)(x)\wedge(p\mu)(y)=0\leq(p\mu)(x-y)$. Therefore, $p\mu$ is a fuzzy subgroup of G.

d) Let $a\in(p\mu)(G)$ be any element. Therefore, $(p\mu)(x)=a$ for some $x\in G$. Now, $(p\mu)(x)=\vee\{\mu(y) \mid x=py, y\in G\}=a$. Since μ has the supremum property, there exists $y\in G$ such that $\mu(y)=a$. Hence $a\in\mu(G)$.Therefore, $p\mu(G)\subseteq\mu(G)$.

Lemma 3 : If μ,ν are fuzzy subgroups of a group G such that μ has supremum property, $\nu\subseteq\mu$, $\mu(e)=\nu(e)$ and ν is neat in μ, then $(p\nu)_a=p\nu_a$ for all a such that $0<a\leq\mu(e)$.

Proof : Since ν is neat in μ, ν_a is neat in μ_a, where $0<a\leq\mu(e)$. Therefore, $p\nu_a=\nu_a\cap p\mu_a$, by definition of neat subgroups. Let $x\in(p\nu)_a$ be any element. Therefore, $(p\nu)(x)\geq a$. We have, $(\nu\cap p\mu)(x)\geq(p\nu)(x)$. This implies $(\nu\cap p\mu)(x)\geq a$. So, $x\in(\nu\cap p\mu)_a$.This implies $x\in\nu_a\cap(p\mu)_a=\nu_a\cap p\mu_a=p\nu_a$. Hence $(p\nu)_a\subseteq p\nu_a$. To prove $p\nu_a\subseteq(p\nu)_a$, suppose $x\in p\nu_a$ be any element. Then, there exists some y in ν_a such that $x=py$. Therefore, $(p\nu)(x)\geq a$. Thus, $x\in(p\nu)_a$. Hence, $p\nu_a\subseteq(p\nu)_a$.

By using the above lemma, we prove the following theorem which gives a characterization of neat fuzzy subgroups.

Theorem 4 : If μ,ν are fuzzy subgroups of a group G such that $\nu\subseteq\mu$, $\nu(e)=\mu(e)$ and μ has the supremum property, then ν is neat μ if, and only if $p\nu=\nu\cap p\mu$ for all positive primes p.

Proof : ν is neat μ $\Leftrightarrow \forall a$ such that $0<a\leq\mu(e)$, ν_a is neat in μ_a

$\Leftrightarrow \forall a$ such that $0<a\leq\mu(e)$, $\forall$ primes p, $p\nu_a=\nu_a\cap p\mu_a$

$\Leftrightarrow \forall a$ such that $0<a\leq\mu(e)$, $\forall$ primes p, $(p\nu)_a=\nu_a\cap(p\mu)_a$

$\Leftrightarrow \forall a$ such that $0 < a \leq \mu(e)$, $\forall$ primes p, $pv_a = (v \cap p\mu)_a$
$\Leftrightarrow \forall$ primes p, $pv = v \cap p\mu.$

Towards the transitive property of neat fuzzy subgroups, we have.

Theorem 5 : If μ, v, γ are fuzzy subgroups of a group G such that $\gamma \subseteq v \subseteq \mu$ and $\mu(e) = v(e) = \gamma(e)$, then γ is neat in v and v is neat in μ implies γ is neat in μ.

Proof : Consider the equation $py_a = x_a$, where $y_a \subseteq \mu$, $x_a \subseteq \gamma$, and p, a prime number. Since $\gamma \subseteq v$ and γ is neat in v, we have $x_a \subseteq v$ and there exists $z_a \subseteq v$ such that $pz_a = x_a$. Since $v \subseteq \mu$ and v is neat in μ, there exists $w_a \subseteq \gamma$ such that $pw_a = x_a$. Hence γ is neat in μ.

The following theorem gives a relation between divisible fuzzy subgroups and neat fuzzy subgroups.

Theorem 6 : If μ, v are fuzzy subgroups of a group G such that $v \subseteq \mu$ and $\mu(e) = v(e)$, then v is divisible implies v is neat in μ. Also, if μ is divisible, then v is neat in μ if, and only if, v is divisible.

Proof : Since v is divisible, v_a is divisible for all a such that $0 < a \leq v(e)$. Since every divisible subgroup is neat, v_a is neat in μ_a for all a such that $0 < a \leq \mu(e)$. Therefore, by theorem 1, v is neat in μ. Since μ is divisible, μ_a is divisible for all a such that $0 < a \leq \mu(e)$. So, $n\mu_a = \mu_a, \forall n \in \mathbb{Z}$.
Now, v is neat in μ $\Leftrightarrow v_a$ is neat in μ_a, for all a such that $0 < a \leq v(e)$.
$\Leftrightarrow pv_a = v_a \cap p\mu_a$, for all a such that $0 < a \leq v(e)$ and for all primes p.
$\Leftrightarrow pv_a = v_a \cap \mu_a$, for all a such that $0 < a \leq v(e)$ and for all primes p, since $p\mu_a = \mu_a$.
$\Leftrightarrow pv_a = v_a$, for all a such that $0 < a \leq v(e)$ and for all primes p.
$\Leftrightarrow v_a$ is divisible for all a such that $0 < a \leq v(e)$.
$\Leftrightarrow v$ is divisible for all a such that $0 < a \leq v(e)$.

Finally, we give an example of a neat fuzzy subgroup.

Example 1: Consider the group $G = \langle a \rangle \oplus \langle b \rangle$ such that $o(a) = p$, $o(b) = p^3$, where p is prime. Let $H = \langle a + pb \rangle$. Then it is easy to see that H is a neat subgroup of G. Define $\mu : G \to [0,1]$ such that $\mu(e) = 1$, $\mu(x) = 1/2$ for $x \in G \setminus \{e\}$ and $v : G \to [0,1]$ such that $v(e) = 1$, $v(x) = 1/2$ for $x \in H \setminus \{e\}$, $v(x) = 1/4$ for $x \in G \setminus H$. Clearly, μ and v are fuzzy subgroups of G since their level sets

$\mu_1 = \{e\}, \mu_{1/2} = G, \nu_1 = \{e\}, \nu_{1/2} = H, \nu_{1/4} = G$ are subgroups of G. Also, $\nu(x) \leq \mu(x)\ \forall x \in G$. Therefore, $\nu \subseteq \mu$ and $\nu_{1/2} = H$ is neat in $\mu_{1/2} = G$. Similarly, it is clear to see that ν_a is neat in μ_a for all a, $0 < a \leq \nu(e) = 1$. Therefore, by Theorem 1, ν is a neat fuzzy subgroup of μ.

REFERENCES

1. Fuchs, L. *Infinite Abelian Groups,* Vol. I, Academic Press, New York (1970).

2. Honda, K. (1956). Realism in the Theory of Abelian Groups I, *Comment. Math.Univ. St. Paul* **5** : 37-75 .

3. Honda,K.(1961). Realism in the Theory of Abelian Groups II, *Comment. Math.Univ. St. Paul* **9**:11-28.

4. Honda,K.(1964). Realism in the Theory of Abelian Groups III, *Comment. Math.Univ. St. Paul* **12** : 75-111.

5. Mordeson,J.N. and Malik,D.S. *Fuzzy Commutative Algebra,* World Scientific, Singapore (1998)

6. Sidky,F.I. and Atif Mishref, M.(1990). Divisible and Pure Fuzzy Subgroups, *Fuzzy Sets and Systems,* **34** : 377-382