STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011-12 & thereafter)

SUBJECT CODE : 11MT/PC/MI24

M. Sc. DEGREE EXAMINATION, APRIL 2013
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : MEASURE THEORY AND INTEGRATION
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

Answer all the questions: **5×2=10**

1. Prove that the outer measure is translation invariant.
2. Show that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.
3. Define: Ring , σ- ring, outer measure.
4. Prove that a countable union of sets positive with respect to a signed measure v is a positive set.
5. Define: Rectangle, measurable rectangle, unit sphere R^k .

SECTION – B

Answer any five questions: **5×6=30**

6. For any sequence of sets, $\{ E_i \}$,prove that $m^*(\cup E_i) \leq \Sigma\, m^* (E_i)$
7. Show that for any set A and any $\varepsilon > 0$, there is an open set O containing A and such that $m^*(O) \leq m^*(A) + \varepsilon$.
8. IF f and g are integrable functions then prove that
   i) αf is integrable and $\int \alpha f dx = \alpha \int f dx$
   ii) $f + g$ is integrable and $\int (f + g) \, dx = \int f \, dx + \int g \, dx$.
   iii) If $f \leq g$ a.e. , then $\int f \, dx \leq \int g \, dx$.
9. Let $f = g \; a.e \; (\mu)$, where μ is a complete measure. Show that if f is measurable, so is g.
10. Show that the following conditions on the signed measures μ & v on [[X , $\mathcal{S}$]] are equivalent:
    i) $v << \mu$ ii) $| v | << | \mu |$ iii) $v^+ << \mu$ iv) $v^- << \mu$
11. Show that there are uncountable sets of measure zero.
12. Prove that the class of elementary sets ξ is an algebra.

SECTION – C

Answer any three questions: **3×20=60**

13. a) Show that every non-empty open set has positive measure.

    b) Let the rationals $\mathbb{Q}$ enumerated as $q_{1,}q_2, q_{3,}$ ... and the set G is defined by

    $G = \bigcup_{n=1}^{\infty}\left(q_n - \frac{1}{n^2},\ q_n + \frac{1}{n^2}\right)$, prove that for any closed set F, $m(G\Delta F) > 0$.

14. a) Let $\{ E_i \}$ be a sequence of measurable sets ,Then prove :

    i) if E_1 contained in E_2 contained in E_3 ,… we have

    $m(\lim E_i) = \lim m(E_i)$.

    ii) If E_1 contains E_2 contains E_3 … & $m(E_i) < \infty$ for each i , then we have

    $m(\lim E_i) = \lim m(E_i)$.

    b) State and prove Lebesgue's Dominated Convergence Theorem.

15. If μ is a measure on a σ – ring $\mathcal{S}$, then the class $\bar{\mathcal{S}}$ of sets of the form $E\Delta N$. For any sets E ,N such that $E \in \mathcal{S}$ while N is contained in some set in $\mathcal{S}$ of zero measure , is a σ- ring , and the set function μ defined by $\mu(E\Delta N) = \mu(E)$ is a complete measure on $\bar{\mathcal{S}}$.

16. Let ν be a signed measure on $[[X , \mathcal{S}]]$ .Then prove that there exists unique measures ν^+ & ν^- on $[[X , \mathcal{S}]]$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

17. Let $[[X, \mathcal{S}, \mu]]$ and $[[Y, \mathfrak{I}, v]]$ be $\sigma -$ finite measure spaces. For $V \in \mathcal{S} \times \mathfrak{I}$ write $\varphi(x) = v(V_x), \Psi(y) = \mu(V^y)$, for each $x \in X, y \in Y$. Then prove that φ us $\mathcal{S} -$ measureable, Ψ is $\mathfrak{I} -$ measureable and $\int_X \varphi d\mu = \int_Y \Psi dv$.

⅄⅄⅄⅄⅄⅄⅄⅄⅄⅄⅄⅄