STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011-12 & thereafter)

SUBJECT CODE : 11MT/PC/LA24

M. Sc. DEGREE EXAMINATION, APRIL 2013
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : LINEAR ALGEBRA
TIME : 3 HOURS **MAX. MARKS : 100**

Section-A
Answer ALL the questions

(5x2=10)

1. Prove that similar matrices have the same characteristic polynomial.
2. Define a cyclic $R-module$ and give an example.
3. If M, of dimension $m,$ is cyclic with respect to T, then prove that the dimension of MT^k $is\ \ m-k$ for all $k \leq m$.
4. State Cayley-Hamilton theorem.
5. Prove that an orthogonal set of non-zero vectors is linearly independent.

Section-B
Answer any FIVE questions **(5x6=30)**

6. Let T be a linear operator on the finite-dimensional space V. Let $c_1, \ldots, c_k$ be the distinct characteristic values of T and let W_i be the space of characteristic vectors associated with the characteristic value c_i. If $W = W_1 + \cdots + W_k$, then prove that $\dim W = \dim W_1 + \cdots \dim W_k$. In fact, if $\mathfrak{B}_i$ is an ordered basis for W_i, then prove that $\mathfrak{B} = (\mathfrak{B}_1, \ldots \mathfrak{B}_k)$ is an ordered basis for W.

7. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$.Prove that T is diaonalizable by exhibiting a basis for R^3, each vector of which is a characteristic vector of T.

8. If A and B are sub-modules of M then prove that $(A+B)/B$ is isomorphic to $A/(A \cap B)$.

9. If $T \in A(V)$ is nilpotent, of index of nilpotence $n_1,$ then prove that a basis of V can be found such that the matrix of T in this basis has the form $\begin{pmatrix} M_{n_1} & 0 & \ldots & 0 \\ 0 & M_{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{n_r} \end{pmatrix}$ where $n_1 \geq n_2 \geq \cdots \geq n_r\ and$ where $n_1 + n_2 + \cdots + n_r = dim_F V$.

10. If T in $A_F(V)$ has as minimal polynomial $p(x) = q(x)^e$, where $q(x)$ is a monic, irreducible polynomial in $F[x]$, then a basis of V over F can be found in which the matrix of T is of the form $\begin{pmatrix} C(q(x)^{e_1}) & & & \\ & C(q(x)^{e_2}) & & \\ & & \ddots & \\ & & & C(q(x)^{e_r}) \end{pmatrix}$ where $e = e_1 \geq e_2 \geq \cdots \geq e_r.$

11. If V is a finite dimensional inner product space and T and U are linear operators on V and if c is scalar then prove that,
    (i) $(T + U)^* = T^* + U^*$
    (ii) $(cT)^* = \bar{c}\, T^*$
    (iii) $(TU)^* = U^*T^*$
    (iv) $(T^*)^* = T$

12. Prove that on a finite-dimensional inner product space of positive dimension, every self-adjoint operator has a non-zero characteristic vector.

Section-C

Answer any THREE questions (3x20=60)

13. a) Let V be a finite dimensional vector space over the field F and let T be a linear operator on V, then prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.
    b) Let T be a linear operator on an n-dimensional vector space over V. Then prove that the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

14. Prove that any finite abelian group is the direct product of cyclic groups.

15. Prove that two nilpotent linear transformations are similar if and only if they have the same invariants.

16. Prove that the elements S and T in $A_F(V)$ are similar in $A_F(V)$ if and only if they have the same elementary divisors and hence or otherwise prove that if two matrices A and B in F_n are similar in K_n where K is an extension of F, then A and B are already similar in F_n.

17. a) If V is a finite dimensional inner product space and f a linear functional on V then prove that there exists a vector β in V such that $f(\alpha) = (\alpha|\beta) \forall\, \alpha \in V$. Also show that if the assumption that V is finite dimensional is dropped then the result fails.
    b) Prove that for every invertible complex $n \times n$ matrix B there exists a unique lower-triangular matrix M with positive entries on the main diagonal such that MB is unitary.