STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011–12)

SUBJECT CODE: 11MT/PC/FA44

M. Sc. DEGREE EXAMINATION, APRIL 2013
BRANCH I – MATHEMATICS
FOURTH SEMESTER

COURSE : CORE
PAPER : FUNCTIONAL ANALYSIS
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION—A (5x2=10)
ANSWER ALL THE QUESTIONS

1. Prove that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$
2. State and prove Schwarz inequality.
3. If T is an operator on H then prove that T is normal if and only if its real and imaginary parts commute.
4. Define the determinant.
5. Define Banach algebra.

SECTION—B (5x6=30)
ANSWER ANY FIVE QUESTIONS

6. State and prove open mapping theorem
7. If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M+N$ is also closed.
8. Let $\{e_1, e_2, ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then prove that $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$; and also prove that $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$ for each j.
9. If P is a projection on H with range M and null space N, then prove that $M \perp N$ if and only if P is self-adjoint and $N = M^{\perp}$.
10. If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other then prove that N_1+N_2 and N_1N_2 are normal.
11. Prove that two matrices in A_n are similar if and only if they are the matrices of a single operator on H relative to different bases.
12. Prove that the set G is regular elements of Banach algebra A is open.

SECTION—C (3x20=60)
ANSWER ANY THREE QUESTIONS

13. State and prove Hahn-Banach theorem.

14. a) Let H be a Hilbert space, and let $\{e_i\}$ be an orthonormal set in H. Prove that the following conditions are all equivalent to one another

(i) $\{e_i\}$ is complete.

(ii) $x \perp \{e_i\} \Rightarrow x = o$

(iii) if x is an arbitrary vector in H, then $x = \sum (x, e_i) e_i$;

(iv) if x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$.

b) If $\{e_i\}$ is an orthonormal set in a Hilbert space H, and if x is an arbitrary vector in H then prove that $x - \sum (x, e_i) e_i \perp e_j$ for each j.

15. a) Prove that the ad joint operation $T \to T^*$ on $\mathrm{B}(H)$ has the following properties:

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$ (ii) $(\alpha T)^* = \alpha\, T^*$ (iii) $T_1 T_2^* = T_2^*\, T_1^*$

$(iv) T^{**} = T$ $(v) \|T^*\| = \|T\|$ $(vi) \|T^* T\| = \|T\|^2$

b) If T is an operator on H for which $(Tx,x)=0$ for all x then prove that $T=0$.

16. State and prove the Spectral theorem.

17. a) If the regular elements of a Banach algebra A are denoted by G then prove that the mapping $x \to x^{-1}$ of G into G is continuous and is a homeomorphism of G onto itself.

b) Prove that the boundary of the set of all singular elements S is a subset of the set all topological divisors of zero Z.