On Random Dynamical Systems and Levels of Their Description

Reinhard Illner and Michèle De La Chevrotière *
Department of Mathematics and Statistics
University of Victoria
Victoria, BC V8W 3P4

January 24, 2012

Abstract

We consider dynamical systems in which a (typically vector-valued) dependent variable evolves according to autonomous dynamics switching randomly according to Markovian laws that change with the value of the dependent variable. Such systems are known as "random evolutions" or, in electrical engineering contexts, as "switching systems". Systems of this type are encountered in applications from electrical engineering to cell biology (our paper was inspired by a recent model for genetic oscillators). We review the derivation of the forward Kolmogorov master equations for the probabilities to find the system in a certain state at some time. In the limit of an infinite switching rate solutions of our system converge almost surely to solutions of an averaged problem. The classical tool of a Chapman-Enskog expansion, well-known from kinetic theory, provides diffusion approximations to first order in the scale parameter. Our work focusses on a few typical examples. The analysis is formal, but we expect that the results hold rigorously in analogy to earlier results from kinetic theory (see [11]).

*Supported by a grant from the Natural Sciences and Engineering Research Council of Canada; AMS Classification 92C37, 60K20, 60K35

A Prologue for Helmut Neunzert

In 1976, as a young man who had just completed the Ph.D., Helmut Neunzert hired me (R.I.) for my first real academic job, at the then still very young University of Kaiserslautern. It was the beginning of 9 years of interaction on kinetic theory and applied mathematics, and of a lifetime friendship. Helmut directed me towards the Boltzmann equation and its mysteries, an area which would define my career.

One of the early things I remember vividly was Helmut's statement "we need to pay more attention to probability theory." As a man of action, he did not just leave it with this statement; he set out to teach intermediate and advanced courses on probability and stochastic processes, courses which I, our visiting colleague Ivan Kuščer and many of our students attended. Helmut had astutely observed that kinetic theory could be approached by using PDEs or Probability, and that there were researchers in either camp, but the communication between these camps was rather tenuous (fortunately, while this was true then, it is not true anymore in 2012). At the biannual Oberwolfach conferences on kinetic theory, Helmut made an effort to invite representatives from either camp, with good long-term results.

I remembered all these things as I began working on modelling problems in cell biology some 8 years ago. Again, probability loomed large in the application we had targeted (namely, an explanation of the possible origin of circadian rhythms at the cellular level; see [4]). While we succeeded, I felt that a more systematic treatment of the context was in order. Such a treatment was provided by Michèle De La Chevrotière, in [2]. (Incidentally, Michèle became a graduate student at the University of Victoria upon a recommendation by none other than Helmut Neunzert.) Since then, the context has become known to us as the theory of random evolutions, and while this theory has been known for several decades [6], its applicability to engineering, biology and finance, and its affinity to the kinetic theory toolbox call for a review of the concept. This is the rationale for this paper.

1 Introduction

The article [4] (see also [7] and [2]) introduced the dynamical system

$$\begin{aligned} R_1' &= k_{11}(1 - I_1) - k_{12}R_1 && (1)\\ P_1' &= k_{13}R_1 - k_{14}P_1 - 2k_{15}P_1^2 + 2k_{16}D_1 && (2)\\ D_1' &= k_{15}P_1^2 - k_{16}D_1 && (3)\\ R_2' &= k_{11}J_1 - k_{12}R_2 && (4)\\ P_2' &= k_{23}R_2 - k_{24}P_2 - 2k_{25}P_2^2 + 2k_{26}D_2 && (5)\\ D_2' &= k_{25}P_2^2 - k_{26}D_2 && (6) \end{aligned}$$

as a model for a transcriptional-translational oscillator (TTO) in cell biology. Here, $X = (R_1, P_1, \ldots, D_2) \in \mathbf{R}_+^6$. The variables R_1 and R_2 are numbers (or fractions) of messenger RNA (mRNA) molecules transcribed by two genes (sites on the DNA strand); they are produced at rates k_{11} while the random variables I_1 and J_1 take values 0 and 1, respectively. R_1 and R_2 have biological half-lives (hence the decay terms), but while they exist they translate raw materials into the proteins P_1 and P_2. These proteins combine further to form homodimers D_1 and D_2. Furthermore, the homodimers attach to the sites where R_1 and R_2 are transcribed, and such that D_2 inhibits the transcription of R_1 while D_1 activates the transcription of R_2. See Figure 1.

This TTO example assumes that the DNA sites transcribe mRNA molecules one at a time while the site is active. The formation of the proteins and homodimers are governed by the laws of mass action, and each of these compounds also has a biological half-life. The parameters $k_{11}, k_{12}, \ldots k_{26}$ determine the rates of these chemical reactions or decay of compounds; see [4] for possible numerical values.

There is no coupling between the equations for R_1, P_1, D_1 and R_2, P_2, D_2 except possibly through the laws of the random variables I_1, J_1, and this is exactly how coupling arises: it is assumed that there are constants λ_0, λ_1 and a scaling parameter $\epsilon > 0$ (to be thought of as small but fixed) such that

$$\begin{aligned} Prob\{I_1 = 0 \quad \text{in } (t, t+h) | I_1(t) = 0\} &= 1 - (\lambda_0 D_2(t)h/\epsilon) + o(h), && (7)\\ Prob\{I_1 = 1 \quad \text{in } (t, t+h) | I_1(t) = 1\} &= 1 - (\lambda_1 h/\epsilon) + o(h) && (8) \end{aligned}$$

and similarly for J_1

$$\begin{aligned} Prob\{J_1 = 0 \quad \text{in } (t, t+h) | J_1(t) = 0\} &= 1 - (\mu_0 D_1(t)h/\epsilon) + o(h), && (9)\\ Prob\{J_1 = 1 \quad \text{in } (t, t+h) | J_1(t) = 1\} &= 1 - (\mu_1 h/\epsilon) + o(h). && (10) \end{aligned}$$

In other words, the switching rates of I_1 from 0 to 1 at time t are $(\lambda_0/\epsilon)D_2(t)$, (because $Prob\{I_1(t+h) = 1 | I_1(t) = 0\} = \lambda_0 D_2(t)h/\epsilon + o(h)$)

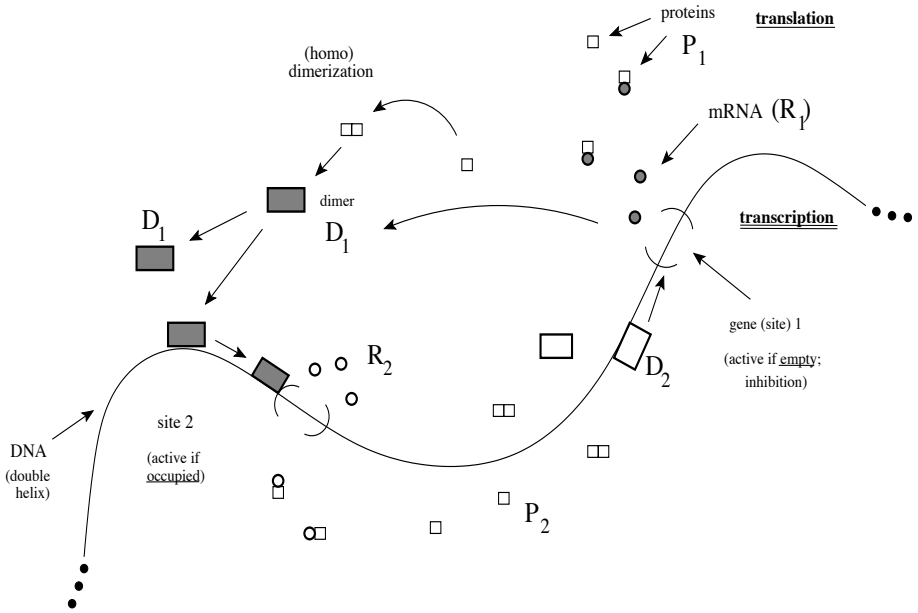

Figure 1: Cartoon of a TTO.

and from 1 to 0, λ_1/ϵ. Again, the parameter ϵ should here be thought of as fixed while h is a variable.

Note that the first rate depends on $D_2(t)$ (consistent with the assumption that the first gene is inhibited by D_2). A similar interpretation applies to the rules governing J_1. We refer to [4] and the references therein for an exposition of the microbiological background. Switching and its significance in gene expression has been discussed via Kolmogorov master equations (and their numerical analysis) in [9].

The parameter ϵ could, of course, be absorbed inside the λs, but we choose not do do so because the purpose of ϵ is a scaling of the switching rates; observe that the quotients of the switching rates (and consequently the quotients between the mean free and occupied times) are independent of ϵ. One of several objectives of this paper is to find ways of describing the behaviour of the expected values and fluctuations of the field variables for small ϵ.

In [4] is was shown that the TTO system does display stable oscillations whose presence can be explained as a Hopf bifurcation. It was further shown how two such TTOs with slightly different periods may couple to produce long-period oscillations that provide a possible explanation of the cellular origin of circadian rhythms. We refer the reader to [4], [7] and [2] for further details.

The TTO example which we have discussed so far is an example of a random evolution dynamical system, in which the time evolution of state variables is governed by a combination of macroscopic laws (for example, laws of mass action) and random variables whose laws are dependent on the state of the system, as shown above. Random evolutions have been known and studied for a long time (see for example [6]). Their application to cell biology and genetic dynamics is a natural consequence of the transition regimes between micro- and mesodynamics relevant in the cell. (for recent related work see [9, 10]).

Random evolutions also arise in electrical engineering applications, where they are often called "switching" systems. See [1] for a recent reference.

The purpose of our paper is to provide a blueprint of how to understand systems of this type at several different levels of description; We begin in full generality but will then focus on a rather simple example to keep the analysis tractable.

The most general scenario of the type of system of interest can be written in the form

$$\dot{X} = f(I_1, I_2, ..., I_k, X) \tag{11}$$

where $X = X(t) \in \mathbf{R}^n$, and I_j switches randomly between m_j values (without restricting the generality, the integers $0, \ldots, m_j - 1$.) At any given time the right-hand side is therefore one of $\Pi_{j=1}^{k} m_j$ functions. In all the examples under consideration we will set $m_j = 2$ for all j, such that I_j can only take the values 0 ("off") or 1 ("on"). We will call this, appropriately, "switching."

In general the laws of the random variables I_j are dependent on the current value of X (or a functional of X), and the switching rates of the random variables, scaled by a parameter $\epsilon > 0$, are very fast. Our paper addresses the following issues.

1. The deterministic limit $\epsilon \searrow 0$.

2. The derivation and discussion of forward Kolmogorov master equations for the systems under consideration, and

3. Diffusion approximations to these master equations, which allow analysis of the time evolution of expected values and fluctuations of X up to second order in ϵ. The tool we use is a Chapman-Enskog expansion, well known from kinetic theory.

It is fair to say that we discuss four different levels of description of the same phenomenon, where each description has its merits at appropriate scales of the switching rates. The first level is, of course, the original problem, for a

given fixed and positive ϵ. For vanishing ϵ one may consider the deterministic limit, leading to a much simpler system of ordinary differential equations (the second level of description). To get a complete understanding of the stochastic dynamics of the system, including information about expected values and fluctuations of X, one derives and analyses (at a third level) the forward Kolmogorov master equations (for a positive ϵ). However, these equations will have exchange terms with a factor $\frac{1}{\epsilon}$, reflecting the high switching rates for small ϵ (in reality, this forces rapid relaxation towards equilibrated states, but it makes the Kolmogorov master equations awkward for use in numerical experiments). Finally, one can use Chapman-Enskog asymptotic expansions to derive drift-diffusion approximations for "macroscopic" densities (in which one no longer keeps track of the particular value of the random variables). This is the fourth level of description.

Our objective in this paper is to present and discuss these levels of description for a few typical examples of (11). While each individual tool we employ is well known, we consider it valuable to present them in unison for these examples.

The presentation should make it transparent that random evolutions are natural mathematical objects in applications from engineering to finance to cell biology, and their analysis draws from a toolbox encompassing probability theory, asymptotic analysis, PDEs and kinetic theory.

1.1 Case studies and well-posedness

The motivating TTO example is a little too complicated for our full program, although the forward Kolmogorov master equations for this case were derived in [2]. Here we will concentrate on the second of the simpler examples

1.
$$\dot{X} = f_I(X), \quad \text{where} \quad X \in \mathbf{R}, \quad I = 0, 1 \tag{12}$$

and

$$Pr\{I(t+h) = 1 | I(t) = 0, X(t) = \xi\} = \frac{1}{\epsilon}\lambda_0(\xi)h + o(h)$$

$$Pr\{I(t+h) = 0 | I(t) = 1, X(t) = \xi\} = \frac{1}{\epsilon}\lambda_1(\xi)h + o(h).$$

The λ_0, λ_1 are assumed to depend smoothly on ξ. We further assume that there are positive constants $0 < C_1 < C_2 < \infty$ such that for all $\xi \quad C_1 \leq \lambda_i(\xi) \leq C_2$.

2.

$$\begin{array}{rcll} \dot{X} & = & f_I(X,Y), & \text{where} \quad (X,Y) \in \mathbf{R}^2, \quad I = 0,1 \\ \dot{Y} & = & g(X,Y), & \text{and} \end{array} \tag{13}$$

$$Pr\{I(t+h) = 1 | I(t) = 0, X(t) = \xi, Y(t) = \eta\} = \frac{1}{\epsilon}\lambda_0(\xi,\eta)h + o(h)$$

$$Pr\{I(t+h) = 0 | I(t) = 1, X(t) = \xi, Y(t) = \eta\} = \frac{1}{\epsilon}\lambda_1(\xi,\eta)h + o(h)$$

We assume (again) that the λ_i depend smoothly on (ξ,η) and are bounded below and above by positive constants. For later reference we denote

$$\boldsymbol{F_0} := (f_0, g), \quad \boldsymbol{F_1} := (f_1, g).$$

The second example differs from the first one in that the law of the random variable I depends on both X and Y, and via a second ODE. This is similar to the situation from the TTO example and is motivated by that example.

Our analysis depends on the solvability of initial value problems associated with the equations under consideration. For simplicity we shall assume that for each possible value of the random variables Problem (11) is globally solvable for all initial data inside a domain of interest, and that bounds uniform in ϵ can be obtained on all these solutions. Specifically, in the context of example 1, let $x_{min}(t)$ and $x_{max}(t)$ be the solutions of the initial value problems

$$\dot{x}(t) = \min_{i=0,1} f_i(x(t)), \quad x(0) = x_0, \tag{14}$$

$$\dot{x}(t) = \max_{i=0,1} f_i(x(t)), \quad x(0) = x_0 \tag{15}$$

and assume that f_0, f_1 are uniformly Lipschitz. Then x_{min} and x_{max} are uniquely and globally defined, and for each ϵ the solution of example 1 for $X(0) = x_0$ satifies $x_{min}(t) \leq X(t) \leq x_{max}(t)$ for all t.

The functions x_{min} and x_{max} are deterministic and do not depend on the laws of the random variable I, and in particular not on ϵ.

1.2 A remark on invariant domains.

In many special cases we can do even better. Suppose that in example 1 we take $f_0(x) = -x$, $f_1(x) = 1 - x$. It is then immediate that the interval $(0,1)$ is invariant under the dynamics: if $0 < X_0 < 1$, then $0 < X(t) < 1$ for all $t > 0$. This has the added advantage that we can assert from the outset that

$f_I(X)$ and $\dot{X}$ will both remain uniformly bounded for all t, regardless of the values of the λ_i or of ϵ.

If we make the minor change to reset $f_0(x) = x$, this invariance property is lost.

The motivating TTO example also has a helpful invariance property: In fact, a careful analysis of the vector field on the right-hand side shows that if $R_1(0) \in (0, k_{11}/k_{12})$ then $R_1(t)$ will stay in this interval, and there is a bounded rectangle in the first quadrant of the $P_1 - D_1$ plane such that for nonnegative initial data the dynamics will keep P_1 and D_1 in that rectangle. Similar properties hold for (R_2, P_2, D_2). It is even true that the dynamics will drive the state of the system into this invariant domain if the initial values are outside.

1.3 Objectives.

The bounds mentioned earlier, or the (stronger) invariance property, if it applies, makes it possible to show that in the limit $\epsilon \to 0$ solutions to our examples converge almost surely to solutions of time-averaged systems that are easily derived from elementary arguments. We discuss this in Section 2. One possible tool to prove this limit rigorously is renewal reward theory; this was done in detail in [4], and we will not repeat the argument here.

In Section 3 we summarize the derivation of the Kolmogorov master equations for our examples. Finally, in Section 4 we apply Chapman-Enskog expansions to derive drift-diffusion approximations. This follows the blueprint from [11], where the procedure was done rigorously for the Carleman model from kinetic theory (for general references to Chapman Enskog expansions in kinetic theory we mention [3] or [8]. Our examples are easier in the sense that the underlying master equations are always linear, and the end result of our analysis is a linear partial differential equation of drift-diffusion type for the evolution of the density distribution function of the field variables. In particular, solution of this equation allows calculation of the expected value and variation of these variables up to second order in ϵ.

2 The averaging limit $\epsilon \to 0$.

Consider our three examples (ex. 1 and 2, and TTO) in the limit $\epsilon \to 0$. In this section we focus on Example 1 and will simply state the results for Example 2 and the TTO model.

Define a *cycle* as a period for which I is first zero (beginning with the switch from 1), then switches to 1. The cycle ends at the next switch back to

0. Suppose that the cycle starts at time t, and suppose that $X(t) = \xi$. The assumptions on the transition rates of I imply that for small ϵ the average time while I stays 0 is $\frac{\epsilon}{\lambda_0(\xi)}$, while the average time while I is 1 is $\frac{\epsilon}{\lambda_1(\xi)}$, up to errors of higher order in ϵ. While $I = 0$ the evolution of X follows f_0, and it switches to f_1 with the switch of I. Formally, therefore, the average rate of change of X during a cycle is approximately (we omit the argument ξ)

$$\frac{\frac{\epsilon}{\lambda_0} f_0 + \frac{\epsilon}{\lambda_1} f_1}{\frac{\epsilon}{\lambda_0} + \frac{\epsilon}{\lambda_1}} = \frac{\lambda_1 f_0 + \lambda_0 f_1}{\lambda_0 + \lambda_1},$$

producing the limit equation

$$\dot{x} = \frac{\lambda_1(x) f_0(x) + \lambda_0(x) f_1(x)}{\lambda_0(x) + \lambda_1(x)}. \tag{16}$$

Note that the right hand side is a convex combination of f_0 and f_1.

For example 2 the same reasoning produces the limit system

$$\dot{x} = \frac{\lambda_1(x,y) f_0(x,y) + \lambda_0(x,y) f_1(x,y)}{\lambda_0(x,y) + \lambda_1(x,y)} \tag{17}$$

$$\dot{y} = g(x,y). \tag{18}$$

Finally, in the TTO example, Eqns. (1) and (4) become

$$\dot{R}_1 = \frac{\lambda_1}{\lambda_0 D_2 + \lambda_1} - k_{12} R_1 \tag{19}$$

$$\dot{R}_2 = \frac{\mu_0 D_1}{\mu_0 D_1 + \mu_1} - k_{12} R_2, \tag{20}$$

and the remaining equations do not change. The following result applies.

Proposition 1. *Suppose that $X(0) = x_0$ for all $\epsilon \geq 0$. For $\epsilon = 0$ we replace the system including stochastic switches by the limit equations derived above. Then, almost surely for all $t > 0$, the solutions of the equations for positive ϵ converge to the solution of the limit equation as $\epsilon \to 0$.*

Proof. For the TTO system this is proved in [4]. The proof given there uses equiboundedness and equicontinuity of the approximating solutions $X = X_\epsilon$, and the renewal reward theorem. This methodology applies to all the cases under consideration.

3 The Kolmogorov master equations

In this section we present the derivation of the Kolmogorov master equations for Example 2. For example 1 we will simply list the result; the TTO example is done in the appendix. We mention in passing that a related system was presented, with some numerical experiments, in [9]. However, no asymptotic analysis for small ϵ was done there.

We denote by T_0 and T_1 the solution operators associated with (13), so that, for example, $\frac{d}{dt}T_0(t)(x,y) = (f_0,g)(T_0(t)(x,y))$. We will further assume that:

1. There exist (sufficiently smooth) probability distribution densities $p_0(\boldsymbol{x},t)$, $p_1(\boldsymbol{x},t)$ such that

$$Pr\{I(t)=i, \boldsymbol{X}(t)\in\Omega\} = \int_\Omega p_i(\boldsymbol{x},t)d\boldsymbol{x}, \qquad i=0,1,$$

where $\Omega = [\xi,\xi+\Delta\xi)\times[\eta,\eta+\Delta\eta)$. We abbreviate $\boldsymbol{\xi}=(\xi,\eta)$, and $O(\Delta) := O(\Delta\xi)+O(\Delta\eta)$.

2. As already stated in the introduction, we assume smooth transition rates λ_0, λ_1 so that

$$Pr\{I(t+h)=1|I(t)=0,\boldsymbol{X}(t)\in\Omega\} = \frac{1}{\epsilon}\big(\lambda_0(\boldsymbol{\xi})+O(\Delta)\big)h+o(h),$$
$$Pr\{I(t+h)=0|I(t)=1,\boldsymbol{X}(t)\in\Omega\} = \frac{1}{\epsilon}\big(\lambda_1(\boldsymbol{\xi})+O(\Delta)\big)h+o(h).$$

(λ_0, λ_1 are smooth in $\boldsymbol{\xi}$, and bounded above and below by positive constants).

This assumption implies in particular that the probability of two subsequent switches in a time interval $[t,t+h]$ is of order $o(h)$. Further, one checks that

$$Pr\{\boldsymbol{X}(t+h)\in\Omega, I(t)=0\} = Pr\{\boldsymbol{X}(t+h)\in\Omega, I(t+h)=0\}+O(h) \quad (21)$$

(because switches are simply of probability $O(h)$). Now expand

$$\begin{aligned} Pr\{\boldsymbol{X}(t+h)\in\Omega, I(t+h)=1\} &= Pr\{\boldsymbol{X}(t+h)\in\Omega, I(t+h)=1, I(t)=0\} \\ &+ Pr\{\boldsymbol{X}(t+h)\in\Omega, I(t+h)=1, I(t)=1\} \\ &= : P_0+P_1 \end{aligned}$$

and we compute further, using (21)

$$\begin{aligned} P_0 &= Pr\{\boldsymbol{X}(t+h)\in\Omega, I(t)=0\}Pr\{I(t+h)=1|I(t)=0,\boldsymbol{X}(t+h)\in\Omega\} \\ &= \Big[Pr\{\boldsymbol{X}(t+h)\in\Omega, I(t+h)=0\}+O(h)\Big] \\ &\times\ Pr\{I(t+h)=1|I(t)=0,\boldsymbol{X}(t+h)\in\Omega\} \\ &= \Big(\int_\Omega p_0(\boldsymbol{x},t+h)d\boldsymbol{x}+O(h)\Big)\Big[\frac{1}{\epsilon}\big(\lambda_0(\boldsymbol{\xi})+O(\Delta)\big)h+o(h)\Big]. \end{aligned}$$

For P_1 we use the fact that no switches have happened (because if I is one at t and at $t+h$, then either 0 or 2,4,6... switches have happened, and the latter event has probability $o(h)$). Hence the dynamics is given by T_1 during the whole time interval, and

$$\begin{aligned} P_1 &= Pr\{I(t+h)=1|I(t)=1,\boldsymbol{X}(t)\in T_1(-h)\Omega\}Pr\{I(t)=1,\boldsymbol{X}(t)\in T_1(-h)\Omega\} \\ &= \left(1-\frac{1}{\epsilon}\lambda_1(T_1(-h)\boldsymbol{\xi})h+o(h)\right)\int_{T_1(-h)\Omega}p_1(\boldsymbol{x},t)d\boldsymbol{x} \\ &= \left(1-\frac{1}{\epsilon}\big(\lambda_1(\boldsymbol{\xi}+O(h))+O(\Delta)\big)h+o(h)\right)\int_{T_1(-h)\Omega}p_1(\boldsymbol{x},t)d\boldsymbol{x} \end{aligned}$$

We set $\boldsymbol{x}=(x,y)=T_1(-h)(\boldsymbol{z})=(z-hf_1(\boldsymbol{z}),u-hg(\boldsymbol{z}))+o(h)$, where $\boldsymbol{z}=(z,u)$. Using this in the obvious substitution we find

$$\int_{T_1(-h)\Omega}p_1(\boldsymbol{x})d\boldsymbol{x} = \int_\Omega p_1(z-hf_1(\boldsymbol{z}),u-hg(\boldsymbol{z}))(1-h(\partial_z f_1(\boldsymbol{z})+\partial_u g(\boldsymbol{z})))\,d\boldsymbol{z}+o(h)$$

Collecting all terms, (22) becomes

$$\begin{aligned} \int_\Omega p_1(\boldsymbol{x},t+h)d\boldsymbol{x} &= \Big(\int_\Omega p_0(\boldsymbol{x},t+h)d\boldsymbol{x}+O(h)\Big)\Big[\frac{1}{\epsilon}\big(\lambda_0(\boldsymbol{\xi})+O(\Delta)\big)h+o(h)\Big] \\ &+ \Big(\int_\Omega p_1(z-hf_1(\boldsymbol{z}),u-hg(\boldsymbol{z}))(1-h(\partial_z f_1(\boldsymbol{z})+\partial_u g(\boldsymbol{z}))\,d\boldsymbol{z}\Big) \\ &\times \Big[1-\frac{1}{\epsilon}\big(\lambda_1(\boldsymbol{\xi}+O(h))+O(\Delta)\big)h+o(h)\Big]. \end{aligned}$$

Dividing by $|\Omega|$ and sending $\Delta\xi$, $\Delta\eta \to 0$ yields

$$\begin{aligned} p_1(\boldsymbol{\xi}, t+h) &= \Big(p_0(\boldsymbol{\xi}, t+h) + O(h)\Big)\Big[\frac{1}{\epsilon}\lambda_0(\boldsymbol{\xi})h + o(h)\Big] \\ &+ p_1(\xi - hf_1(\boldsymbol{\xi}), \eta - hg(\boldsymbol{\xi}))\Big(1 - h(\partial_\xi f(\boldsymbol{\xi}) + \partial_\eta g(\boldsymbol{\xi}))\Big) \\ &\times \Big[1 - \frac{1}{\epsilon}\big(\lambda_1(\boldsymbol{\xi} + O(h))\big)h + o(h)\Big] \\ &= \frac{1}{\epsilon}\lambda_0(\boldsymbol{\xi})p_0(\boldsymbol{\xi}, t+h)h + p_1(\xi - hf_1(\boldsymbol{\xi}), \eta - hg(\boldsymbol{\xi})) \\ &\times \Big[1 - \frac{1}{\epsilon}\lambda_1(\boldsymbol{\xi} + O(h))h - h\big(\partial_\xi f_1(\boldsymbol{\xi}) + \partial_\eta g(\boldsymbol{\xi})\big)\Big] + o(h). \end{aligned}$$

Subtracting $p_1(\boldsymbol{\xi}, t)$ from both sides, dividing by h and taking the limit $h \to 0$ gives

$$\begin{aligned} \lim_{h\to 0}\frac{1}{h}\Big[p_1(\boldsymbol{\xi}, t+h) - p_1(\boldsymbol{\xi}, t)\Big] &= \lim_{h\to 0}\frac{1}{h}\Big[p_1(\xi - hf_1(\boldsymbol{\xi}), \eta - hg(\boldsymbol{\xi})) - p_1(\boldsymbol{\xi}, t)\Big] \\ &- p_1(\boldsymbol{\xi})\Big[\big(\partial_\xi f_1(\boldsymbol{\xi}) + \partial_\eta g(\boldsymbol{\xi})\big)\Big] + \frac{1}{\epsilon}(\lambda_0(\boldsymbol{\xi})p_0(\boldsymbol{\xi}, t) - \lambda_1(\boldsymbol{\xi})p_1(\boldsymbol{\xi})) \end{aligned}$$

or

$$\begin{aligned} \frac{\partial}{\partial t}p_1(\boldsymbol{\xi}, t) &= -\Big(f_1, g\Big)\cdot\nabla p_1(\boldsymbol{\xi}, t) - p_1(\boldsymbol{\xi})\Big[\big(\partial_\xi f_1 + \partial_\eta g\big)\Big] \\ &+ \frac{1}{\epsilon}(\lambda_0(\boldsymbol{\xi})p_0(\boldsymbol{\xi}, t) - p_1(\boldsymbol{\xi})\lambda_1(\boldsymbol{\xi})). \end{aligned}$$

An analogous calculation applies for p_0. Summarizing, these probability densities satisfy the forward Kolmogorov master equations

$$\begin{aligned} \frac{\partial p_0}{\partial t} + \nabla_{\boldsymbol{x}}\cdot\Big(\boldsymbol{F_0}p_0\Big) &= \frac{1}{\epsilon}(\lambda_1 p_1 - \lambda_0 p_0). \\ \frac{\partial p_1}{\partial t} + \nabla_{\boldsymbol{x}}\cdot\Big(\boldsymbol{F_1}p_1\Big) &= \frac{1}{\epsilon}(\lambda_0 p_0 - \lambda_1 p_1). \end{aligned}$$

The corresponding calculation for Example 1 leads to the Kolmogorov master equations

$$\begin{aligned} \partial_t p_1 + \partial_x(f_1 p_1) &= \frac{1}{\epsilon}\Big[\lambda_0 p_0 - \lambda_1 p_1\Big] \\ \partial_t p_0 + \partial_x(f_0 p_0) &= \frac{1}{\epsilon}\Big[\lambda_1 p_1 - \lambda_0 p_0\Big] \end{aligned} \tag{22}$$

These Kolmogorov master equations intrinsically contain all the information about the system we could only ask for. The probability densities p_0 and p_1 can in principle be computed for all times from initial data, and they allow the calculation of all the relevant quantities like expectations and variance of the components of X. For an ϵ of order 1, this is feasible, and some work in this sense has been done in [9]. However, for small ϵ the numerical solution of these equations becomes unreasonable; and even if it could be done, it may provide information the user is not really interested in. After all, it may not be of interest whether I at a particular moment takes the value 0 or 1; what matters in practical applications is the cumulative (diffusive) effect of the switches of I on X.

One idea one might pursue is to study the moment equations derived from the Kolmogorov master equations: for example 1, one multiplies the system equations by powers x^k where $k = 0, 1, 2, \ldots$, integrates over x and uses boundary conditions if necessary. This procedure leads to recursively solvable systems of ODEs if the rates λ_i are constant, and if the functions f_i are simple enough (for example, for $f_i(x) = i - x$.) This was demonstrated in [7]. However, for the generality we are considering in the current work the emerging system of moment equations is not a closed system and therefore not solvable.

A powerful alternative is based on a suitable asymptotic expansion in ϵ, which allows to reduce the system to one equation of drift-diffusion type. We demonstrate this reduction in the next section.

4 The Chapman-Enskog expansion

We restrict the discussion to Example 2. As stated previously, the Kolmogorov master equations contain all the necessary information about the system. However, the exchange terms on the right-hand side carry a factor $\frac{1}{\epsilon}$, which makes them highly oscillatory for small ϵ. It is intuitively expected that the exchange terms lead to diffusive behaviour at a macroscopic level, and this intuition is borne out by mathematical analysis. This is what we are going to do in this section: our objective is to find a diffusion-type PDE for the evolution of $\rho(\boldsymbol{x}, t) := p_0(\boldsymbol{x}, t) + p_1(\boldsymbol{x}, t)$, accurate at 1st order in ϵ.

We recall the Kolmogorov master equations

$$\partial_t p_i(\boldsymbol{x}, t) + \mathrm{div}_{\boldsymbol{x}} \left[p_i(\boldsymbol{x}, t) \begin{pmatrix} f_i(\boldsymbol{x}) \\ g(\boldsymbol{x}) \end{pmatrix} \right] = \frac{1}{\epsilon} \Big[\lambda_k(\boldsymbol{x}) p_k(\boldsymbol{x}, t) - \lambda_i(\boldsymbol{x}) p_i(\boldsymbol{x}, t) \Big],$$

where $k = i + 1 \bmod 2$. It is convenient to set

$$\boldsymbol{m} = \begin{pmatrix} p_0 f_0 + p_1 f_1 \\ (p_0 + p_1) g \end{pmatrix}, \quad d := \frac{1}{\lambda_0 + \lambda_1}(\lambda_0 p_0 - \lambda_1 p_1).$$

and we abbreviate further

$$s_0 := \frac{\lambda_0}{\lambda_0 + \lambda_1}, \qquad s_1 := \frac{\lambda_1}{\lambda_0 + \lambda_1}.$$

Then $d = s_0 p_0 - s_1 p_1$. In the new variables ρ and d the system is equivalent to

$$\partial_t \rho + \mathrm{div}_{\boldsymbol{x}} \boldsymbol{m} = 0,$$

and

$$\partial_t d + s_0 \,\mathrm{div}_{\boldsymbol{x}} \left[p_0 \begin{pmatrix} f_0 \\ g \end{pmatrix} \right] - s_1 \mathrm{div}_{\boldsymbol{x}} \left[p_1 \begin{pmatrix} f_1 \\ g \end{pmatrix} \right] = \frac{1}{\epsilon}(\lambda_1 p_1 - \lambda_0 p_0) = -\frac{\lambda_0 + \lambda_1}{\epsilon} d.$$

Here ρ is the density function of the unkown X; d is an exchange term which relaxes rapidly (at rate $\frac{1}{\epsilon}$), and $\boldsymbol{m}$ can be expressed as a combination of ρ and d. The objective of the Chapman-Enskog expansion is to eliminate the 2nd equation and to express $\boldsymbol{m}$ in terms of ρ alone; the expansion cannot hold on an initial layer of length $O(1/\epsilon)$, but provides an approximation (to order $o(\epsilon)$) to $p_0 + p_1$ on time intervals bounded away from 0. So we have two original unknowns p_0, p_1 and wish to reduce our model to just one target unknown ρ. To this end, we first express $\boldsymbol{m}$ in terms of ρ and d. Specifically, we write

$$\boldsymbol{m} = A \begin{pmatrix} \rho \\ d \end{pmatrix}, \quad \text{with} \quad A = A(\boldsymbol{x}).$$

To find the entries of the matrix A, we first express p_0 and p_1 in terms of ρ and d :

$$\left.\begin{array}{c} p_0 + p_1 = \rho \\ s_0 p_0 - s_1 p_1 = d \end{array}\right\} \Rightarrow \begin{array}{rcl} p_0 &=& d + s_1 \rho \\ p_1 &=& -d + s_0 \rho \end{array}$$

Using this in the definition of $\boldsymbol{m}$ and

$$\boldsymbol{m} = \begin{pmatrix} p_0 f_0 + p_1 f_1 \\ (p_0 + p_1) g \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} \rho \\ d \end{pmatrix}.$$

we easily compute

$$\begin{array}{ll} \alpha_1 = s_0 f_1 + s_1 f_0, & \beta_1 = f_0 - f_1, \\ \alpha_2 = g, & \beta_2 = 0. \end{array}$$

The equation for ρ is thus

$$\partial_t \rho + \mathrm{div}_{\boldsymbol{x}} \left[A \begin{pmatrix} \rho \\ d \end{pmatrix} \right]. \tag{23}$$

and the equation for d is

$$\partial_t d + s_0 \,\mathrm{div}_{\boldsymbol{x}} \Big[(d + s_1\rho) \begin{pmatrix} f_0 \\ g \end{pmatrix} \Big] - s_1 \mathrm{div}_{\boldsymbol{x}} \Big[(-d + s_0\rho) \begin{pmatrix} f_1 \\ g \end{pmatrix} \Big] = -\frac{1}{\epsilon}(\lambda_0 + \lambda_1) d,$$

The key idea of the Chapman-Enskog expansion is to expand d in a power series in ϵ (but not to expand the target variable ρ). This is motivated by the expectation of rapid decay of d.

We therefore write $d = \sum_{k=0}^{\infty} \epsilon^k d_k$ and wish to use the first order approximation $d_0 + \epsilon d_1$ in (23). From the equation for d:

$$\begin{aligned} \sum_{k=0}^{\infty} \epsilon^k (d_k)_t \;+\; & s_0 \left(f_0 \Big(\sum_{k=0}^{\infty} \epsilon^k d_k + s_1 \rho \Big) \right)_x \\ +\; & s_0 \left(g \Big(\sum_{k=0}^{\infty} \epsilon^k d_k + s_1 \rho \Big) \right)_y \\ -\; & s_1 \left(f_1 \Big(-\sum_{k=0}^{\infty} \epsilon^k d_k + s_0 \rho \Big) \right)_x \\ -\; & s_1 \left(g \Big(-\sum_{k=0}^{\infty} \epsilon^k d_k + s_0 \rho \Big) \right)_y \\ =\; & -\frac{1}{\epsilon}(\lambda_0 + \lambda_1) \sum_{k=0}^{\infty} \epsilon^k d_k. \end{aligned}$$

Matching coefficients of equal powers of ϵ we find $d_0 = 0$ and thus $\partial_t d_0 = 0$, so that we can discard d_0. Then match coefficients of ϵ^0:

$$s_0(f_0 s_1 \rho)_x + s_0(g s_1 \rho)_y - s_1(f_1 s_0 \rho)_x - s_1(g s_0 \rho)_y = -(\lambda_0 + \lambda_1) d_1$$

This determines d_1. We write out the equation (23) for ρ:

$$\partial_t\rho + \partial_x\Big((s_0f_1 + s_1f_0)\rho + (f_0 - f_1)d)\Big) + \partial_y(g\rho) = 0,$$

and substitute ϵd_1 for d, where we neglect terms of order $o(\epsilon)$ and d_1 is given by

$$\begin{aligned} d_1 &= \frac{\lambda_1}{(\lambda_0+\lambda_1)^2}\left[\left(\frac{\lambda_0}{\lambda_0+\lambda_1}f_1\rho\right)_x + \left(\frac{\lambda_0}{\lambda_0+\lambda_1}g\rho\right)_y\right] \\ &- \frac{\lambda_0}{(\lambda_0+\lambda_1)^2}\left[\left(\frac{\lambda_1}{\lambda_0+\lambda_1}f_0\rho\right)_x + \left(\frac{\lambda_1}{\lambda_0+\lambda_1}g\rho\right)_y\right] \end{aligned}$$

(we have resubstituted $s_0 = \dfrac{\lambda_0}{\lambda_0+\lambda_1}$, etc.) Writing $d \approx d_0 + \epsilon d_1$ in (23) produces

$$\begin{aligned} \partial_t\rho + \partial_x\Bigg(&\frac{\lambda_0f_1+\lambda_1f_0}{\lambda_0+\lambda_1}\rho - \epsilon\frac{\lambda_0(f_0-f_1)}{(\lambda_0+\lambda_1)^2}\left[\Big(\frac{\lambda_1}{\lambda_0+\lambda_1}f_0\rho\Big)_x + \Big(\frac{\lambda_1}{\lambda_0+\lambda_1}g\rho\Big)_y\right] \\ &+\epsilon\frac{\lambda_1(f_0-f_1)}{(\lambda_0+\lambda_1)^2}\left[\Big(\frac{\lambda_0}{\lambda_0+\lambda_1}f_1\rho\Big)_x + \Big(\frac{\lambda_0}{\lambda_0+\lambda_1}g\rho\Big)_y\right]\Bigg) + \partial_y(g\rho) = 0. \end{aligned} \tag{24}$$

Equation (24) is our target equation. It is not immediately clear that it is an equation of drift-diffusion type, but by calculating the coefficients C_{xx}, C_{xy}, C_{yy} of $\rho_{xx}, \rho_{xy}, \rho_{yy}$, we find:

$$C_{yy} = 0$$

(this is not surprising because there is no switching in the equation for Y)

$$C_{xx} = -\epsilon\frac{(f_0-f_1)^2\lambda_0\lambda_1}{(\lambda_0+\lambda_1)^3} \le 0,$$

and

$$C_{xy} = 0.$$

If we set $\epsilon = 0$ in (24) we are left with the partial differential equation

$$\partial_t\rho + \partial_x\Big(\frac{\lambda_0f_1+\lambda_1f_0}{\lambda_0+\lambda_1}\rho\Big) + \partial_y(g\rho) = 0. \tag{25}$$

The characteristic equations visible in this PDE of transport type are the limit equations from section 2, as they should be. This is a consistency test of our theory.

5 Conclusions

We showed how a dynamical system in which the right-hand sides switch rapidly according to Markovian random switches whose laws depend on the system itself can be analysed at various levels of description- in its original form, in the limit of infinite rapid switching rates (leading to a system of ODEs), via the associated Kolmogorov master equations, and finally via a drift-diffusion approximation obtained by a Chapman-Enskog expansion. Our discussion was formal, but given that the original systems admit strong a priori bounds on the values and rates of change of the dependent variables, and that the computed drift-diffusion equations obtained via the CE expansion are linear, we assert that the approximation of $p_0 + p_1$ by ρ is accurate to order ϵ^2 on time intervals $[\epsilon, T]$, where an initial layer must be exlcuded (see [12],[11]).

In applications, the drift-diffusion equation may be used to compute expected values $\int x\rho(x,t)\,dx$ or fluctuations of the dependent variable accurate to second order in ϵ. Finally, it is an interesting question whether the intrinsic diffusivity in biological systems incorprating switches (such as cells) carries an adaptive advantage. The Chapman-Enskog expansion should also allow to distinguish intrinsic diffusion (as discussed here) from external noise.

Acknowledgement. This research was supported by an NSERC Discovery grant. Reinhard Illner would like to express his gratitude to the Department of Aeronautical Engineering at the University of Kyoto, where some of the calculations were performed during a stimulating visit.

References

[1] Boukas, E.-K.: Stochastic switching systems: analysis and design. Birkhäuser, Boston (2006)

[2] De La Chevrotière, M.: On a jump Markovian model for a gene regulatory network. M.Sc. thesis, University of Victoria (2008)

[3] Cercignani, C.: Theory and Application of the Boltzmann equation. Springer-Verlag (1989).

[4] Edwards, R., Gibson, R., Illner, R., and Paetkau, V.: A stochastic model for circadian rhythms from coupled ultradian oscillators. Theor. Biol. Med. Mod. **4**(1), 2007 (http://www.tbiomed.com/content/4/1/1)

[5] Edwards, R., Illner, R., and Paetkau, V.: A model for generating circadian rhythm by coupling ultradian oscillators. Theor. Biol. Med. Mod.**3**(12), 2006 (http://www.tbiomed.com/content/3/1/12)

[6] Griego, R., Hersh, R.: Theory of random evolutions with applications to partial differential equations. Trans. AMS **156**, 1971

[7] Illner, R.: Modeling with dynamical systems and kinetic equations. Riv. Mat. Univ. Parma **6**(7), 199-243 (2007)

[8] Kremer, G. M.: An Introduction to the Boltzmann equation and transport processes in gases. Springer-Verlag Berlin-Heidelberg (2010).

[9] Lipniacki, T., et al.: Transcriptional stochasticity in gene expression. J. of Theor. Biol. **238**, 348-367 (2006)

[10] Mackey, M.C., Tyran-Kamińska, M., and Yvinec, R.: Molecular distributions in gene regulatory dynamics. J. of Theor. Biol. **274**, 84-96 (2011)

[11] Palczewski, A.: Exact and Chapman-Enskog solutions for the Carleman model. Math. Meth. Appl. Sci. **6**, 417-432 (1984)

[12] O'Malley, R.E.: Introduction to Singular Perturbations. Academic Press (1974).