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ON RATIONALIZATION OF THE FOUNDATIONS OF DIFFERENTIAL CALCULUS

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ABSTRACT

Logical analysis of differential calculus for the purpose of rationalization of the foundations of mathematics is proposed. Methodological basis for analysis is the unity of formal logic and rational dialectics. It is shown that the standard definitions of the concepts "variable quantity", "infinitesimal (infinitely diminishing) quantity", and "infinitely large (infinitely increasing) quantity" represent logical errors because these definitions are based on the concept of mathematical process. The correct definition of the concept "variable quantity" and the unified mathematical definition relating to both extreme cases of values of variable quantity (i.e. to the case of arbitrarily small values of the variable quantity and the case of arbitrarily large values of the variable quantity) and the case of arbitrary values of the variable quantity are proposed.

Key words: Differential calculus, Mathematical analysis, Foundation of mathematics, Philosophy of mathematics .

INTRODUCTION

As is known, the confidence in the scientific method of research and in rational thinking replaced all other ways of cognition in the 20th century. Rational thinking represents the greatest achievement of mankind. Rationalization of thinking and of science is dialectical imperative of our time. The development of rational thinking in the 21st century leads to critical analysis, reconsideration, and rationalization of the generally accepted theories created by the classics of science (for example, physicists N. Bohr, E. Schrödinger, W. Heisenberg, mathematicians I. Newton, G. Leibniz, L. Euler, J. Lagrange, A. Cauchy, etc.). Rationalization and critical analysis of science are two side pieces (component factors) in progress of science. Critical analysis and rationalization of theories are based on formal-logical analysis of scientific concepts, of the completeness of concepts, of the completeness of a system of concepts because "only the completeness leads to clarity" (Confucius).

Recently, independent researchers give attention to critical analysis of theoretical physics, mathematics, biology, etc. (see, for example, www.gsjournal.net). In the process of critical analysis and interpretation of scientific theories, "...we can hardly rely on any of the old principles even if they are very common. The only mandatory requirement is the absence of logical contradictions" (N. Bohr). Logical consistency of theories is achieved with use of the formal-logical laws. And a natural-scientific interpretation of theories is based on the use of rational dialectics. System of universal (general-scientific) concepts and laws – i.e., science of the general laws of development of the Nature, human society, and correct thinking – is the unity of formal logic and rational dialectics. This unity is not only correct methodological basis of science but also the correct methodological basis for a critical analysis of theories.

The concept of measure represents methodological (philosophical, dialectical) key for critical analysis of mathematical relations. The measure is a philosophical category (general-scientific concept) designating the unity of qualitative and quantitative determinacy (aspects) of a material object. The measure expresses belonging of quantitative determinacy (quantitative aspect) to qualitative determinacy (qualitative aspect). Quantitative determinacy is studied by pure and applied mathematics. Pure mathematics studies the quantitative determinacy abstracted from the qualitative determinacy of object. Therefore, pure mathematics have no any natural-scientific (for example, physical) meaning: "understanding of mathematics as ideal formalism elucidates the question of truth in mathematics. The difficulty of it is that the ideal objects of mathematics not only are not confronted with reality but have no sufficient exact prototype in the latter" [1]. Applied mathematics – as mathematical formalism of science – studies the measure of the material object. Therefore, any mathematical relation of natural sciences must have natural-scientific meaning and must belong to material object (i.e. it must contain reference to material object). Quantitative determinacy of the property of the material object is called the natural-scientific quantity (for example, physical quantity) and represents the measure of the material object.

Qualitative and quantitative determinacy of material object obeys the formal-logical laws. Therefore, in accordance with the law of identity, the left-hand side and right-hand side of the mathematical relation must belong to the same material object. According to the law of contradiction, the left-hand and right-hand sides of the mathematical relation must not belong to different qualitative determinacy of the object.

Today, there are no works devoted the critical analysis of differential calculus within the framework of the unity of formal logic and rational dialectics. The purpose of this work is to propose formal-logical analysis of foundations of differential calculus [2-4] for rationalization of the standard foundations of mathematics. The methodological basis of the work is the unity of formal logic and rational dialectics.

1. Definitions of the standard mathematical concepts

1. If variable quantity x tends to limit a, this mathematical process is written symbolically in the form $x \to a$ or in the form $\lim x = a$.

Definition of the concept "limit of variable quantity". Variable quantity x tends to limit a if $|x-a|$, starting with some instant of time, will become and will always remain less than any preassigned positive number $\varepsilon << 1$, however small it may be, i.e., if the inequality

$$|x-a|<\varepsilon \ ,$$

starting with some instant of time, will always true under any $\varepsilon << 1$. Parameter ε characterizes the range of variable quantity x.

2. If variable quantity x tends to limit 0, this mathematical process is written symbolically in the form $x \to 0$ or in the form $\lim x = 0$.

Definition of the concept "infinitesimal". The variable x is called infinitesimal if $|x|$, starting with some instant of time, will become and will always remain less than any preassigned positive number $\varepsilon << 1$, however small it may be:

$$|x| < \varepsilon \, .$$

No constant number, however small it may be, is never infinitesimal. The exception to all the numbers is zero which is considered to be the infinitely small quantity, despite the fact that zero is a constant number.

3. There are three symbols $+\infty$, $-\infty$, ∞ which are called "infinity". Every number, rational or irrational, is inherently always "finite". There is no infinite numbers. The role of the symbols $+\infty$, $-\infty$, ∞ is that they characterize a behavior of variables which are "finite" at any moment of time.

Definition of concept "infinitely large quantity (infinities $+\infty$, $-\infty$, ∞)". The variable x is called a positive infinitely large quantity (i.e. positive infinity) if its change will have the following character: with time, it will become and will continue always to be less than any arbitrarily chosen positive number a, however large a may be, i.e. if the inequality $x > a$ is realized at some moment of time and then remains always true. In this case, they say that x "increases without limit". This is symbolically designated in the form of the conditional equality $\lim x = +\infty$. Infinity ∞ is not a limit in true sense (i.e., as they have defined it) because true limit is a number but infinity is not a number. Both infinitely large quantity and infinitely small quantity are above all the variable quantities which have quantitative (numerical, finite) values at any moment of time.

4. Relation between infinitely large x and infinitely small α is as follows. Unit divided by infinitely large x is an infinitesimal; unit divided by infinitesimal α which has never been vanished is an infinitely large x:

$$\alpha = \frac{1}{x}, \quad x = \frac{1}{\alpha}; \quad \lim_{x\to\infty} \alpha = \lim_{x\to\infty} \frac{1}{x} = \frac{\lim\limits_{x\to\infty} 1}{\lim\limits_{x\to\infty} x} = \frac{1}{\infty} = 0 \, .$$

This statement is written in the form of the following conditional equalities:

$$\frac{1}{\infty} = 0, \quad \frac{1}{0} = \infty \, .$$

If there is quotient $\frac{\alpha}{\alpha}$ where the numerator and the denominator are equal infinitely small quantities, then $\frac{\alpha}{\alpha} = 1$ is not infinitesimal. From this it follows that

$$\frac{\alpha}{\alpha} = \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{x}{x} = 1 .$$

Example:

$$\lim_{x\to\infty} \frac{x+2}{3x} = \lim_{x\to\infty} \frac{1+\frac{2}{x}}{3} = \frac{1+\frac{2}{\infty}}{3} = \frac{1+0}{3} = \frac{1}{3}.$$

The range of values of the variable x is determined by the following relation: $0 < x < \infty$.

5. Quantity x and increment Δx are independent variable quantities. Therefore, derivative is a function of x.

2. INCORRECT PROPOSITIONS UNDERLYING FOUNDATIONS OF MATHEMATICAL ANALYSIS

The standard foundations of mathematical analysis are based on the following incorrect propositions:

1. The operation $\frac{p}{\infty}$ (where $p \neq 0$ is some number) has mathematical (quantitative) sense because the expression $\frac{p}{\infty}$ possesses the numerical value 0:

$$\frac{p}{\infty} = 0 .$$

This expression has the sense of the logical law of identity:

(numerical value) = (numerical value).

Practically, this implies that the symbol ∞ in the standard course of mathematical analysis has mathematical (quantitative) meaning and the operation $\frac{p}{0}$ is considered to be permissible. In addition, the symbol ∞ in the designations of the integral $\int_0^\infty$ and the sum $\sum_{i=0}^{i=\infty}$ have also a quantitative meaning.

2. If the range of values of the variable x is determined by the relation $0 < x < \infty$, then this relation signifies that symbol ∞ and the number 0 have

identical (quantitative) senses. Really, the number 0 (i.e. value of quantities x) and the symbol ∞ one can compare with each other if only they have the same sense.

3. Mathematical definitions relating to the two extreme cases of a variable quantity (i.e. to the case of an infinitesimal variable and to the case of an infinitely large variable) are different.

4. Quantity x and increment Δx are independent variable quantities. Therefore, derivative is function of x.

3. THE PROPOSITION OF FORMAL LOGIC AND RATIONAL DIALECTICS WHICH MUST UNDERLIE THE CORRECT FOUNDATIONS OF MATHEMATICS

The correct foundations of mathematical analysis must be based on the following propositions of formal logic and rational dialectics:

1. A material object represents a unity of qualitative and quantitative determinacy (aspects). This unity is designated with the philosophical concept "measure". Qualitative and quantitative determinacy (aspects) do not exist separately from a material object. Quantitative determinacy (as a result of measurement of the property of the object) belongs to the qualitative determinacy (i.e., to the property) of the object. The qualitative determinacy of the material object is characterized by the concepts "exist" or "not exist". If a qualitative determinacy exists, then a quantitative determinacy too. If a qualitative determinacy does not exist, then a quantitative determinacy does not exist too.

2. Quantitative determinacy (as a result of the measurement of the property of the material object) has a lower and upper boundaries and is expressed with numbers. Number 0 (zero) represents the origin (i.e. lower boundary) of quantitative determinacy. Therefore, the number 0 is not infinitesimal. The number 0 does not mean and does not designate absence of qualitative determinacy of material object.

3. From geometrical point of view, the number 0 represents the point for which there is no segment. From physical point of view, the number 0 is not a measure. Therefore, formal logic prohibits values of measurable quantity divide by 0. In other words, the division operation of any number $p \neq 0$ by the number 0 have no quantitative (mathematical) meaning, and, therefore, it is an inadmissible operation. The inadmissible operation $\frac{p}{0}$ of division of any number $p \neq 0$ by the number 0 is designated with the symbol ∞:

$$\frac{p}{0} = (inadmissible\ \ operation) = \infty .$$

The symbol ∞ does not possess any numerical values, i.e. the symbol ∞ has no mathematical (quantitative) meaning. The symbol ∞ has only logical meaning of the prohibition (i.e., conventional meaning) because it designates the inadmissibility of the operation $\frac{p}{0}$

4. Mathematical definition relating to the two extreme cases of the value of variable quantity (i.e., to the case of arbitrarily small values of variable x and to the case of arbitrarily large values of variable x) must be unified one. One can formulate the following universal mathematical definition that covers all cases of values of variable.

Definition. The variable x is called arbitrarily close to number a if

$$|x - a| < \varepsilon << 1 \ ,$$

where a is an arbitrary number, $\varepsilon << 1$ is an arbitrarily small positive number (parameter) that characterizes the range of the variable x. The variable x is called arbitrarily small (i.e. arbitrarily close to zero) if $a = 0$. The variable x is called arbitrarily large (i.e. arbitrarily far from zero) if $a >> 1$ is an arbitrarily large.

Then, in compliance with this definition, integral $\int_0^a$ and sum $\sum_{i=0}^{i=N}$ have the correct form in the case of $a >> 1$ and natural number $N >> 1$.

5. Quantity x and increment Δx are not independent variable quantities. These quantities must be in the following relation: $x + \Delta x = c$ where $c = \text{const}$. Therefore, derivative of function $f(x)$ must be a function of constant c [5, 6]. Then the following crucial question arises: May one consider this constant c as a variable quantity?

4. DISCUSSION

The idea of motion and the problems of the mechanics had essential impact on the origin and development of mathematical analysis. The idea of motion gave rise to the mathematical concepts "variable", "infinitesimal (infinitely diminished) quantity", and "infinitely large (infinitely increasing) quantity". This important circumstance should be taken in consideration under criticism of the foundations of mathematics.

As is known from the standard course of differential calculus [2-4], the quantity x which with time t change its numerical value is called variable quantity. That number which designates the letter x at moment of time t is called the value of variable quantity $x(t)$ at this moment of time t. Generally speaking, this value changes from moment to moment. This fact is expressed by the words "the letter x runs through set of values". The process of changing values is called mathematical process and is designated with the symbol "$\rightarrow$". Therefore, the standard definition of variable leads to the following point of view on the mathematical formalism: the mathematical formalism must include the mathematical process (i.e. the process of change). This viewpoint is expressed with the concepts of "infinitesimal (infinitely diminished) quantity" and "infinitely large (infinitely increasing) quantity".

However, the standard viewpoint [2-4] is incorrect. Indeed, in viewpoint of formal logic and of rational dialectics, mathematics as a system of statements does not contain any mathematical process (for example, the process of change of numerical values of quantity x). The mathematical formalism is not a material device (as, for example, computer) which carries out change of values of mathematical quantity. Therefore, the standard definition of variable quantity should be replaced with the following correct definition: quantity x that takes different numerical values is called variable; the change of numerical values does not represent a mathematical process. A set of numerical values is called the tolerance range (range of admissible values) of the variable. A character (an order) of change of numerical values is determined by the desired conditions. This definition negates the concept of the mathematical process. Consequently, this definition means that the standard definitions of concepts "infinitesimal (infinitely diminished) quantity" and "infinitely large (infinitely increasing) quantity" represent logical errors because based on the concept "mathematical process".

CONCLUSION

Thus, the critical analysis of the foundations of differential calculus leads to the following conclusion:

(a) mathematical formalism must not be based on the concept "mathematical process" which is expressed with the symbol "$\rightarrow$";

(b) the standard definitions of concepts "variable", "infinitesimal (infinitely diminished) quantity, and "infinitely large (infinitely increasing) quantity" represent logical errors because these definitions are based on the concept "mathematical process";

(c) the standard definitions should be replaced by new, correct definition of variable quantity (which negates the concept of mathematical process) and the unified, correct mathematical definition in the common case of any values of variable.

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