**STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086**
**(For candidates admitted from the academic year 2009-10 & thereafter)**

**SUBJECT CODE : MT/PC/DG44**

**M. Sc. DEGREE EXAMINATION, APRIL 2012**
**BRANCH I – MATHEMATICS**
**SECOND SEMESTER**

**COURSE : CORE**
**PAPER : DIFFERENTIAL GEOMETRY**
**TIME : 3 HOURS** **MAX. MARKS : 100**

**SECTION – A**

**ANSWER ANY FIVE QUESTIONS :** **(5 X 8 = 40)**

1. Show that a unit speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion is a circle.
2. Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Show that its curvature is $k = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}$.
3. Define Mobius band and find its standard unit normal. Show that Mobius band is not orientable.
4. Obtain the expression for the first fundamental form of a surface patch $\sigma$. find the first fundamental form of the plane $\sigma(u, v) = a + up + vq$.
5. Show that the area of a surface patch is unchanged by reparametrisation.
6. State and prove Euler's theorem.
7. Find the Gaussion curvature of the surface of revolution $\sigma(u, v) = (f(u)\cos v, f(u)\sin v\,, g(u))$ where $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$ everywhere.

**SECTION – B**

**ANSWER ANY THREE QUESTIONS :** **(3 X20 = 60)**

8. a) Let $\gamma(s)$ and $\bar{\gamma}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same curvature $k(s) > 0$ and the same torsion $\tau(s)$ for all $s$. Show that there is a rigid motion $M$ of $\mathbb{R}^3$ such that $\bar{\gamma}(s) = M(\gamma(s))$ for all $s$. Also show that if $k$ and $t$ are smooth functions with $k > 0$ everywhere, then there is a unit-speed curve in $\mathbb{R}^3$ whose curvature is $k$ and whose torsion is $t$.

   b) Find the curvature and torsion of the curve $\gamma(t) = \left(a(1 + \cos t), a\sin t, 2a\sin\frac{1}{2}\,t\right)$.

(12+8)

9. a) Let $\sigma: U \to \mathbb{R}^3$ be a patch of a surface $S$ containing a point $P$ of $S$, and let $(u, v)$be coordinates in $U$. show that the tangent space to $S$ at $P$ is the vector subspace of $\mathbb{R}^3$ spanned by the vectors $\sigma_u$ and $\sigma_v$ ( the derivatives are evaluated at the point $(u_0, v_0) \in U$ such that $\sigma(u_0, v_0) = P$.

b) Show that the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 1\}$ is a surface.

(8+12)

10. Show that a diffeomorphism $f: S_1 \to S_2$ is conformal if and only if, for any surface patch $\sigma_1$ on $S_1$, the first fundamental forms of $\sigma_1$ and $f \circ \sigma_1$ are proportional.

11. a) Let $k_1$ and $k_2$ be the principle curvatures at a point $P$ of a surface patch $\sigma$. Show that

(i) $k_1$ and $k_2$ are real numbers.

(ii) if $k_1 = k_2 = k$, say, then $\mathcal{F}_{11} = k\mathcal{F}_1$ and (hence) every tangent vector to $\sigma$ at $P$ is a principle vector.

(iii) if $k_1 \neq k_2$, then any two (non-zero) principle vectors $t_1$ and $t_2$ corresponding to $k_1$ and $k_2$ respectively, are perpendicular.

b) If $\gamma(t) = \sigma(u(t), v(t))$ is a unit-speed curve on a surface patch $\sigma$, show that its normal curvature is given by $k_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$, where $Ldu^2 + 2Mdudv + Ndv^2$ is the second fundamental form of $\sigma$.

(16+4)

12. State and prove Gauss's Remarkable theorem. (20)