STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2011-12)

SUBJECT CODE : 11MT/PC/MI24
M. Sc. DEGREE EXAMINATION, APRIL 2012
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : MEASURE THEORY AND INTEGRATION
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

Answer all the questions: **5×2=10**

1. Define: Lebesgue outer measure.
2. If ψ is a measurable function, then prove $\int_{A\cup B} \psi\, dx = \int_A \psi\, dx + \int_B \psi\, dx$, for any two disjoint measurable sets A and B.
3. When is a set function μ a measure?
4. Define $L^{\infty}(\mu)$.
5. Define the Jordan decomposition of a signed measure.

SECTION – B

Answer any five questions: **5×6=30**

6. If $\{Ei\}$ is a sequence of measurable sets, then prove that if
   (i) $E_1 \subseteq E_2 \subseteq E_3$ .... then $m(\lim E_i) = \lim\ m(E_i)$.
   (ii) $E_1 \supseteq E_2 \supseteq E_3$ .... & $m(E_i) < \infty$ for each i, then $m(\lim E_i) = \lim\ m(E_i)$.
7. For any sequence of sets (E_i), prove that $m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$
8. State and prove Lebesgue's Monotone Convergence Theorem.
9. Let $[[X, \mathcal{S}, \mu]]$ be a measure space and f be a non-negative measurable function. Then prove $\varphi(E) = \int f d\mu$ is a measure on the measurable space $\left[[X, \mathcal{S}]\right]$. If in addition $\int f d\mu < \infty$, then prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if $A \in \mathcal{S}$, and $\mu(A) < \delta$, then $\varphi(A) < \varepsilon$.
10. Show that if $\varphi(E) = \int f d\mu$ where $\int f d\mu$ is defined, then φ is a signed measure.
11. Prove that a countable union of sets positive with respect to a signed measure ν is a positive set.
12. If $E \in \mathcal{S} \times \mathfrak{I}$, then for each, $x \in X$, and $y \in Y$, prove that $E_x \in \mathfrak{I}$ and $E^y \in \mathcal{S}$.

SECTION – C

Answer any three questions: **3×20=60**

13. (i) The rationals Q are enumerated as $q_1, q_2, q_3, \ldots$ and the set G is defined by

$G = \bigcup_{n=1}^{\infty}\left(q_n - \frac{1}{n^2},\ q_n + \frac{1}{n^2}\right)$. Prove that for any closed set $F, m(G \Delta F) > 0$.

(ii) Show that there exists uncountable sets of zero measure.

14. State and prove Fatou's Lemma.

15. If $1 \leq p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $\|f_n - f_m\|_p \longrightarrow 0$ as $n, m \longrightarrow \infty$, then prove that there exists a function f and a subsequence $\{n_i\}$ such that $\lim_{n\to\infty} f_{n_i} = f\ a.e.$ Also prove that $f \in L^p(\mu)$ and $\lim_{n\to\infty}\|f_n - f\|_p = 0$.

16. a) Discuss Hahn decomposition of a measureable space $[[X, \mathcal{S}]]$ with signed measure υ.

b) State and prove Lebesgue Decomposition Theorem.

17. a) Let f be a non-negative $\mathcal{S} \times \mathfrak{I}$ measureable function and let $\varphi(x) = \int_Y f_x\ d\nu$, $\psi(y) = \int_X f^y\ d\mu$ for each $x \in X,\ y \in Y$ then prove that φ is $\mathcal{S}-$ measurable, ψ is $\mathfrak{I}-$ measurable and $\int_X \varphi\ d\mu = \int_{X\times Y} f\ d(\mu \times \nu) = \int_Y \psi\ d\nu$.

b) State and prove Fubini's Theorem.