STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI – 86
(For candidates admitted from the academic year 2023 – 2024 and thereafter)

M.Sc. DEGREE EXAMINATION, APRIL 2026
BRANCH I - MATHEMATICS
SECOND SEMESTER

COURSE	**:**	**CORE**
PAPER	**:**	**MEASURE THEORY AND INTEGRATION**
SUBJECT CODE	**:**	**23MT/PC/MI24**
TIME	**:**	**3 HOURS** **MAX. MARKS: 100**

Q. No.	SECTION A ($\mathbf{5 \times 2 = 10}$) Answer ALL questions	CO	KL
1.	Define a regular measure on $\mathbb{R}$.	1	1
2.	Give an example of an integrable function on $[0, 1]$.	1	1
3.	State whether every algebra is a ring.	1	1
4.	What is meant by a signed measure.	1	1
5.	Mention the x-section and y-section of a set E.	1	1
Q. No.	**SECTION B ($\mathbf{10 \times 1 = 10}$) Answer ALL questions**	**CO**	**KL**
6.	The outer measure of the set $\{x \in \mathbb{R}: \sin x = 0\}$ is 0 (b) 1 (c) 2 (d) ∞	2	2
7.	Which one of the following statements is correct? (a) Every measurable set is open. (b) Every bounded set is measurable. (c) Every set of finite outer measure is measurable. (d) Every open set is an F_σ-set.	2	2
8.	If f is measurable and g is continuous, then $g\ o\ f$ is (a) continuous (b) differentiable (c) measurable (d) constant	2	2
9.	Which of the following is NOT necessarily a Borel set in $\mathbb{R}$? (a) Open interval (b) Closed interval (c) Countable intersection of open sets (d) Lebesgue measurable set.	2	2
10.	Let $\mathcal{M}$ be the class of all Lebesgue measurable subsets of $\mathbb{R}$. Then $\mathcal{M}$ is (a) a ring but not a σ-algebra (b) an algebra but not a σ-algebra (c) a σ-algebra (d) only a hereditary class.	2	2
11.	Which convergence theorem requires the existence of an integrable dominating function? (a) Monotone Convergence Theorem (b) Fatou's Lemma (c) Dominated Convergence Theorem. (d) Fubini's Theorem	2	2

12.	A measurable set A is called a positive set with respect to a signed measure ν if (a) $\nu(A) = 0$. (b) $\nu(E) \geq 0$ for every measurable subset $E \subset A$ (c) A is finite (d) A is open.	2	2
13.	The Radon–Nikodym derivative $\frac{d\nu}{d\mu}$ is (a) always continuous (b) always bounded (c) a measurable function (d) a constant	2	2
14.	Under which condition is the existence of a product measure ensured for two measure spaces? (a) When both measures are bounded (b) When both measures are σ-finite (c) When both spaces are finite (d) When both measures are continuous.	2	2
15.	Which of the following sets of conditions characterizes a probability measure P on a probability space $(\Omega, \mathcal{F}, P)$ according to Kolmogorov's axioms? (a) $P(A) \geq 0$ for every event A, $P(\Omega) = 1$, and P is countably additive on pairwise disjoint events (b) $P(A) \leq 1$ for every event A, $P(\emptyset) = 1$, and P is finitely additive (c) $P(A) \geq 0, P(\Omega) = 0$, and $P(A \cup B) = P(A) + P(B)$ for all events (d) $P(A)$ is bounded, $P(\emptyset) = 0$, and P is multiplicative on disjoint events.	2	2
Q. No.	**SECTION C ($\mathbf{2 \times 15 = 30}$)** **Answer ANY TWO questions**	**CO**	**KL**
16.	Let $E \subset \mathbb{R}$. Apply the regularity properties of Lebesgue outer measure to establish the equivalence of following statements: a) E is measurable. b) For all $\epsilon > 0$, there exists an open set, $O \supseteq E$ such that $m^*(O - E) \leq \epsilon$. c) There exists G, a G_δ-set, $G \supseteq E$ such that $m^*(G - E) = 0$. d) For all $\epsilon > 0$, there exists a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \epsilon$. e) There exists F, a F_σ-set, $F \subseteq E$ such that $m^*(E - F) = 0$.	3	3

17.	Let f be a bounded function on a finite interval $[a, b]$. Verify the equality between the Riemann integral of f and its corresponding Lebesgue integral whenever the Riemann integral exists.	3	3
18.	Let f be a measurable function and $A = \{x: f(x) \geq 0\}$. Using the properties of measurable functions and applying the basic results on integrals of non-negative functions, show that for $c > 0$, $\mu(\{x: f(x) > c\}) \leq \int_A c^{-1} f \, d\mu$.	3	3
19.	Prove that a signed measure ν defined on a measurable space $[\![X, \mathcal{S}]\!]$ decomposes the set X into a disjoint union of a positive set and a negative set. Also, determine the uniqueness of the decomposition.	3	3
Q. No.	**SECTION D** $(\mathbf{2 \times 15 = 30})$ **Answer ANY TWO questions**	**CO**	**KL**
20.	Let $\mathcal{M}$ be the class of Lebesgue measurable sets. Prove that $\mathcal{M}$ is a σ-algebra.	4	4
21.	Let $\{f_n\}$ be a sequence of non-negative measurable functions. Analyze the behavior of $\liminf f_n$ and prove that $\int \liminf\limits_{n\to\infty} f_n \, dx \leq \liminf\limits_{n\to\infty} \int f_n \, dx$.	4	4
22.	Justify the Randon-Nikodym Theorem.	4	4
23.	(i) Examine the statement: If $E \in \mathcal{S} \times \mathcal{T}$, then $E_x \in \mathcal{T}, E^y \in \mathcal{S}$ for $x \in X, y \in Y$. (5 marks) (ii) Let $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}, (x, y) \neq (0,0)$. Show that $\int_0^1 dx \int_0^1 f(x, y) \, dy = \frac{\pi}{4}$ and $\int_0^1 dy \int_0^1 f(x, y) \, dx = -\frac{\pi}{4}$. Also, explain why the results differ, even though the domain is the same. (10 marks)	4	4
Q. No.	**SECTION E** $(\mathbf{2 \times 10 = 20})$ **Answer ANY TWO questions**	**CO**	**KL**
24.	Verify with reasoning: There exists a non-measurable subset of $\mathbb{R}$.	5	5

25.	If μ is a measure on a σ-ring $\mathcal{S}$, $\bar{\mathcal{S}} = \{E \,\Delta\, N : E \in \mathcal{S}, N \subseteq M$ for some $M \in \mathcal{S}$ with $\mu(M) = 0\}$ and define $\bar{\mu}(E \,\Delta\, N) = \mu(E)$. Validate the following statements: a) $\bar{\mathcal{S}}$ is a σ-ring. b) $\bar{\mu}$ is a complete measure on $\bar{\mathcal{S}}$	5	5
26.	Let ν be a signed measure on a measurable space $[\![X, S]\!]$. Analyze how ν can be written as $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are mutually singular measures and the decomposition is unique.	5	5
27.	Let X be a random variable and $M_X(t)$ be finite for all $\|t\| < \epsilon$, for some $\epsilon > 0$. Substantiate the following properties: a) $E\|X\|^n < \infty$ for all $n \geq 1$. b) $M_X(t) = \sum_{n=0}^{\infty} t^n \frac{\mu_n}{n!}$ for all $\|t\| < \epsilon$. $M_X(.)$ is infinitely differentiable on $(-\epsilon, \epsilon)$ and for $r \in \mathbb{N}$, the r^{th} derivative of $M_X(.)$ is $M_X^{(r)}(t) = \sum_{n=0}^{\infty} t^n \frac{\mu_{n+r}}{n!} = E(e^{tX} X^r)$ for all $\|t\| < \epsilon$.	5	5

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