STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2023 – 24 onwards)

M. Sc. DEGREE EXAMINATION, NOVEMBER 2024
BRANCH I - MATHEMATICS
THIRD SEMESTER

COURSE	:	**CORE**
PAPER	:	**FUNCTIONAL ANALYSIS**
SUBJECT CODE	:	**23MT/PC/FA34**
TIME	:	**3 HOURS**

MAX. MARKS : 100

Q. No.	SECTION A $(5 \times 2 = 10)$ Answer ALL questions	CO	KL
1.	Discuss the different norms constructed on K^n.	1	1
2.	Define a continuous seminorm on a Banach space *X*.	1	1
3.	Define dual of a normed space.	1	1
4.	What are orthonormal polynomials? Classify them.	1	1
5.	What are normal and unitary operators?	1	1

Q. No.	SECTION B $(10 \times 1 = 10)$ Answer ALL questions	CO	KL
6.	A normed space is __________. (a) linear space (b) metric space (c) both linear and metric space (d) None of the above	2	2
7.	The set of all continuous functions with compact support on a metric space T is denoted as _______. (a) $C(T)$ (b) $C_{cs}(T)$ c) $C_c(T)$ (d) $C_0(T)$	2	2
8.	If X is an infinite dimensional normed space, then it contains a hyperspace which is ________. (a) compact (b) complete (c) closed (d) not closed	2	2
9.	The set $\{x_n\}$ where $x_n \in C[0,1]$ is _____ at each point of [0,1]. (a)bounded (b) uniformly bounded (c) not bounded (d) bounded but not uniformly bounded	2	2
10.	If *Z* is a closed subspace of a normed space *X*, then the quotient map from *X* to *X*/*Z* is ______. (a) open (b) continuous (c) open and continuous (d) closed	2	2
11.	If X and Y are normed spaces and $F \in BL(X, Y)$, then (a) $\|F\| \neq \|F'\| \neq \|F''\|$ (b) $\|F\| = \|F'\| \neq \|F''\|$ (c) $\|F\| \neq \|F'\| = \|F''\|$ (d) $\|F\| = \|F'\| = \|F''\|$	2	2

12.	The closed unit ball of the dual of a separable normed space is (a) weakly compact (b) sequentially compact (c) compact (d) weak* sequentially compact	2	2
13.	If $A \in BL(H)$then A is self-adjoint if ______. (a) $A^* = -A$ (b) $A^* = A$ (c) $A^* = -A^{-1}$ (d) $A^* = A^{-1}$	2	2
14.	A self-adjoint operator A on a Hilbert space H is said to be positive-definite if $\langle A(x), x \rangle$ is ______ for every nonzero $x \in H$. (a) equal to 0 (b) less than 0 (c) greater than 0 (d) greater than or equal to 0	2	2
15.	If X is a Banach space over $\boldsymbol{K}$ and $A \in BL(X)$ then $\sigma(A)$ is ______ subset of $\boldsymbol{K}$. (a) bounded (b) closed (c) compact (d) All of the above	2	2

Q. No.	**SECTION C ($2 \times 15 = 30$)** **Answer ANY TWO questions**	**CO**	**KL**
16.	a) State and prove Riesz Lemma. b) Prove that the interior and closure of a convex subset E of a normed space X is convex. **(9+6)**	3	3
17.	State and prove closed graph theorem.	3	3
18.	a) Prove: Let X be a normed space. Then i) Let X_0 be a dense subspace of X. For $x' \in X'$, let $F(x')$ denote the restriction of x' to X_0. Then the map F is a linear isometry from X'onto ${X_0}'$. ii) If X' is separable, then so is X. b) State and prove Polarization identity. **(10+5)**	3	3
19.	Let H be a Hilbert space and $A \in BL(H)$. Then prove the following: a) Let A be self-adjoint. Then $\|A\| = \sup\{\|\langle A(x), x \rangle\|; x \in H, \|x\| \leq 1\}$. In particular, $A = 0$ if and only if $\langle A(x), x \rangle = 0$ for all $x \in H$. b) A is unitary if and only if $\|A(x)\| = \|x\|$ for all $x \in H$ and A is surjective. c) A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$. In that case that $\|A^2\| = \|A^*A\| = \|A\|^2$.	3	3

Q. No.	SECTION D $(2 \times 15 = 30)$ **Answer ANY TWO questions**	CO	KL
20.	a) Let Y be a closed subspace of a normed space X. For $x + Y$ in the quotient space X/Y, let $\|\|\|x + Y\|\|\| = \inf\{\|\|x + y\|\|: y \in Y\}$. Prove that $\|\|\|\square\|\|\|$ is a norm on X/Y. b) State and prove Jensen's inequality. **(10+5)**	4	4
21.	If X is a normed space and $A \in BL(X)$, then prove that A is invertible if and only if A is bounded below and surjective. Also, if X is a Banach space then prove that i) A is invertible if and only if A is bijective ii) A is invertible if and only if A is bounded below and the range of A is dense in X.	4	4
22.	a) If X is an inner product space and $f \in X'$ and if $\{u_1, u_2, ...\}$ is an orthonormal set in X, then prove that $\sum_n \|f(u_n)\|^2 \leq \|\|f\|\|^2$. b) State and prove Riesz representation theorem. **(3+12)**	4	4
23.	a) Let H be a Hilbert space and $A \in BL(H)$. Then prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle A(x), y\rangle = \langle x, B(y)\rangle$. b) Let H be a Hilbert bspace and let $A, B \in BL(H)$ and $k \in K$. Then prove that i) $(A + B)^* = A^* + B^*$; ii) $(AB)^* = B^*A^*$; iii) $(kA)^* = \bar{k}A^*$. **(9+6)**	4	4
Q. No.	**SECTION E** $(2 \times 10 = 20)$ **Answer ANY TWO questions**	**CO**	**KL**
24.	State and prove Hahn-Banach separation theorem.	5	5
25.	Prove: i) Let X be a normed space and E be a subset of X. Then E is bounded in X if and only if $f(E)$ is bounded in K for every $f \in X'$. ii) Let X and Y be normed spaces and $F: X \to Y$ be linear. Then F is continuous if and only if $g \circ F$ is continuous for every $g \in Y'$.	5	5
26.	a) Establish Schur's lemma. b) If X is a separable normed space, then prove that every bounded sequence in X' has a weak* convergent subsequence. **(5+5)**	5	5
27.	State and prove Projection theorem.	5	5