STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI – 86
(For candidates admitted from the academic year 2023 – 2024 and thereafter)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2024
BRANCH I - MATHEMATICS
FIRST SEMESTER

COURSE	**:**	**MAJOR CORE**	
PAPER	**:**	**ABSTRACT ALGEBRA**	
SUBJECT CODE	**:**	**23MT/PC/AA14**	
TIME	**:**	**3 HOURS**	**MAX. MARKS: 100**

Q. No.	SECTION A ($5 \times 2 = 10$) Answer ALL questions	CO	KL
1.	Find the number of non-isomorphic abelian groups of order 200.	1	1
2.	Prove that a Euclidean ring possesses a unit element.	1	1
3.	State Eisenstein criterion.	1	1
4.	Prove that the ring $R[x]$, of all polynomials in x over an integral domain R is an integral domain.	1	1
5.	Express the polynomial $x_1^2 + x_2^2 + x_3^2$ in the elementary symmetric functions in x_1, x_2, x_3.	1	1

Q. No.	SECTION B ($10 \times 1 = 10$) Answer ALL questions	CO	KL
6.	If a group G has an order of 18, what can you conclude about the existence of elements of order 3 in G? a) There are no elements of order 3 b) There is exactly one element of order 3 c) There are at least two elements of order 3 d) There are infinitely many elements of order 3	2	2
7.	Let G be a group of order 4 and H be a group of order 9. What can be said about the direct product $G \times H$? a) The order of $G \times H$ is 13. b) The order of $G \times H$ is 36. c) The order of $G \times H$ is 36, and it is a cyclic group. d) The order of $G \times H$ is 36, and it is an abelian group	2	2
8.	Which type of groups are guaranteed to have a unique decomposition into cyclic subgroups? a) Finite non-abelian groups b) Infinite non-abelian groups c) Finite abelian groups d) Infinite abelian groups	2	2
9.	Which rings are characterized by the existence of a division algorithm? a) Euclidean rings b) Unique factorization rings c) Polynomial rings d) Field rings	2	2
10.	What do polynomial rings over commutative rings generalize? a) Division rings b) Integral domains c) Fields d) Euclidean rings	2	2

11.	Which theorem states that every positive integer can be written as a sum of four squares? a) Euclidean theorem b) Unique Factorization theorem c) Polynomial theorem d) Fermat's theorem	2	2
12.	In the context of field extensions, if F is a field and E is an extension field of F, which of the following statements is true? a) E is a proper subset of F b) E contains elements not in F c) E is equal to F d) E has no elements in common with F	2	2
13.	What is the splitting field of a polynomial? a) A field in which the polynomial has no roots b) The smallest field extension of F in which the polynomial can be factored completely c) A field in which the polynomial can be divided by another polynomial d) A field in which the polynomial can be written as a linear combination of other polynomials	2	2
14.	In Galois theory, what do the elements of the Galois group correspond to? a) Roots of polynomials b) Automorphisms of field extensions c) Subfields of extension fields d) Euclidean rings	2	2
15.	What is the key to determining if a polynomial is solvable by radicals? a) The degree of the polynomial b) The number of real roots c) The coefficients of the polynomial d) The Galois group of the polynomial	2	2

Q. No.	**SECTION C ($2 \times 15 = 30$)** **Answer ANY TWO questions**	CO	KL
16.	Let G be a group and suppose that G is the internal direct product of $N_1, N_2, \dots, N_n$ and $T = N_1 \times N_2 \times \dots \times N_n$. Prove that G and T are isomorphic. Provide a clear and rigorous explanation with the necessary results	3	3
17.	Investigate that $J[i]$ is a Euclidean ring. Find the greatest common divisor of $11 + 7i$ and $18 - i$ in $J[i]$.	3	3
18.	(a) Let $\mathbb{R}$ be the field of real numbers and $\mathbb{Q}$ the field of rational numbers. Prove that $\sqrt{2}$ and $\sqrt{3}$ are both algebraic over $\mathbb{Q}$. Also exhibit the polynomial of degree 4 over $\mathbb{Q}$ satisfied by $\sqrt{2} + \sqrt{3}$. (b) If 'a' is any algebraic number, then prove that there is a positive integer 'n' such that na is an algebraic integer. (7+8)	3	3
19.	Prove that K is a normal extension of F if and only if K is the splitting field of some polynomial over F.	3	3

Q. No.	**SECTION D** ($\mathbf{2 \times 15 = 30}$) **Answer ANY TWO questions**	**CO**	**KL**
20.	Prove that if G is a group of finite order p, a prime number and $p^m \mid o(G), p^{m+1} \nmid o(G)$, then G has a p −Sylow subgroup.	4	4
21.	If R is a unique factorization domain, prove that the polynomial ring $R[x]$ is also a unique factorization domain, by proving all the necessary results that are used in the proof.	4	4
22.	Let τ be an isomorphism of a field F onto a field F' defined by $\alpha\tau = \alpha', \forall\ \alpha \in F$. Prove that there is an isomorphism τ^* of $F[x]$ onto $F'[t]$ with the property that $\alpha\tau^* = \alpha', \forall\ \alpha \in F$.	4	4
23.	Let F be a field and let $F(x_1, x_2, \dots, x_n)$ be the field of rational functions in $x_1, x_2, \dots, x_n$ over F. Suppose S is a field of symmetric rational functions, then prove that (i) $[F(x_1, x_2, \dots, x_n): S] = n!$ (ii) $G(F(x_1, x_2, \dots, x_n), S) = S_n$, the symmetric group of degree n (iii) If $a_1, a_2, \dots, a_n$ are the elementary symmetric functions in $x_1, x_2, \dots, x_n$, then $S = F(a_1, a_2, \dots, a_n)$ (iv) $F(x_1, x_2, \dots, x_n)$ is the splitting field over $F(a_1, a_2, \dots, a_n) = S$ of the polynomial $t^n - a_1 t^{n-1} + a_2 t^{n-2} - \cdots + (-1)^n a_n$.	4	4

Q. No.	**SECTION E** ($\mathbf{2 \times 10 = 20}$) **Answer ANY TWO questions**	**CO**	**KL**
24.	Prove that two abelian groups of order p^n are isomorphic if and only if they have the same invariants.	5	5
25.	Prove that every non-zero element in a Euclidean ring R can be uniquely written (up to associates) as a product of prime elements or is a unit in R (prove the necessary results).	5	5
26.	Prove that if F is of characteristic 0 and if a, b are algebraic over F, then there exists an element $c \in F(a, b)$ such that $F(a, b) = F(c)$.	5	5
27.	Prove that the polynomial $f\ (x)\ \in\ F[x]$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a non-trivial common factor.	5	5

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