STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2008–09 & thereafter)

SUBJECT CODE : MT/MC/AS54

B. Sc. DEGREE EXAMINATION, NOVEMBER 2012
BRANCH I - MATHEMATICS
FIFTH SEMESTER

COURSE : MAJOR – CORE
PAPER : ALGEBRAIC STRUCTURES
TIME : 3 HOURS **MAX. MARKS : 100**

ANSWER ANY SIX QUESTIONS

1. a) Define an equivalence relation with an example.

   b) Prove that the distinct equivalence classes of an equivalence on a set A decomposes A as a union of mutually disjoint subsets and conversely.

   c) If H and K are subgroups of a group G, prove that $H \cap K$ is also a subgroup of G.

   (5+7+5)

2. a) If G is a finite group whose order is a prime number p, then prove that G is a cyclic group.

   b) If H and K are finite subgroups of a group G, of orders $O(H)$ and $O(K)$ respectively, prove that $O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$.

   c) Prove that N is a normal subgroup of G if and only if $gNg^{-1} = N$ for every $g \in G$.

   (4+9+4)

3. a) If φ is a homomorphism of G into G, prove that

   (i) $\varphi(e) = \bar{e}$, the unit element of $\bar{G}$.

   (ii) $\varphi(x^{-1}) = [\varphi(x)]^{-1}$ for all $x \in G$.

   b) State and prove the fundamental theorem of homomorphism.

   c) If φ is a homomorphism of G into $\bar{G}$ with kernel K, prove that K is a normal subgroup of G.

   (4+8+5)

4. a) State and prove Cayley's theorem.

   b) Define inner automorphism. Prove that $\mathfrak{I}(G) \approx G/Z$ where $\mathfrak{I}(G)$ is the group of inner automorphisms of G and Z is the center of G.

   c) Obtain the orbits and cycles of the permutation.

   $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

   (7+6+4)

5. a) Prove that every finite integral domain is a field.
   b) If p is a prime number, prove that J_p, the ring of integers mod p is a field.
   c) If F is a field, prove that its only ideals are $\{0\}$ and F itself.

(7+5+5)

6. a) Prove that the intersection of two ideals of a ring R is an ideal of R.
   b) If R is a commutative ring with unit element whose only ideals are (0) and R itself, then prove that R is a field.
   c) Prove that the homomorphism φ of R into R' is an isomorphism if and only if $I(\varphi) = 0$ where $I(\varphi)$ is the kernel of φ.

(3+7+7)

7. a) Define a maximal ideal. If R is a commutative ring with unit element and M is an ideal of R, then prove that M is a maximal ideal of R if and only if R/M is a field.
   b) Prove that any homomorphism of a field is either an isomorphism or a zero map.
   c) Prove that every field is a principal ideal domain.

(10+4+3)

8. a) Show that every integral domain can be embedded in a field.
   b) State and prove the division algorithm for polynomials. (12+5)

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