STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI – 86
(For candidates admitted from the academic year 2023 – 2024)

M.Sc. DEGREE EXAMINATION, APRIL 2024
BRANCH I - MATHEMATICS
SECOND SEMESTER

COURSE	**:**	**MAJOR CORE**
PAPER	**:**	**MEASURE THEORY AND INTEGRATION**
SUBJECT CODE	**:**	**23MT/PC/MI24**
TIME	**:**	**3 HOURS**

MAX. MARKS: 100

Q. No.	SECTION A ($5 \times 2 = 10$) Answer ALL questions	CO	KL
1.	Define Borel sets.	1	1
2.	Define measurable function.	1	1
3.	Define outer measure and give an example.	1	1
4.	When do you say that a signed measure ν on $[\![X, \mathcal{S}]\!]$ is σ-finite?	1	1
5.	State Cauchy-Schwarz inequality.	1	1

Q. No.	SECTION B ($10 \times 1 = 10$) Answer ALL questions	CO	KL
6.	For any set A there exists a measurable set E containing A and such that (a) $m^*(A) = m(E)$ (b) $m^*(A) \neq m(E)$ (c) $m^*(A) > m(E)$ (d) none of these	2	2
7.	Let $[I_n]$ be a finite set of intervals covering the rationals in $[0,1]$. Then $\sum l(I_n)$ is (a) 0 (b) 1 (c) ≥ 1 (d) ≤ 1	2	2
8.	A measurable function f is essentially bounded if (a) $ess\ \sup\lvert f\rvert > \infty$ (b) $ess\ \sup\lvert f\rvert < \infty$ (c) $ess\ \sup\lvert f\rvert = \infty$ (d) $ess\ \sup\lvert f\rvert \neq \infty$	2	2
9.	Which of the following is not true for integrable function f (a) $f = 0$ a.e. $\Rightarrow \int f\,dx = 0$ (b) if $f \leq g$ a.e. $\Rightarrow \int f\,dx \leq \int g\,dx$ (c) f is finite-valued a.e. (d) None of the above	2	2
10.	A measure μ on $\mathcal{R}$ is σ- finite if, (a) $\mu(E_n) \to \infty$ (b) $\mu(E_n) = \infty$ (c) $\mu(E_n) < \infty$ (d) $\mu(E_n) > \infty$	2	2
11.	Which of the following is not true? (a) The class of finite unions of intervals of the form $[a, b)$ forms a ring. (b) Every algebra is a ring (c) Every σ- algebra is a σ-ring (d) Every σ- ring is σ-algebra	2	2
12.	If two measures ν_1 and ν_2 are mutually singular then we denote it by (a)$\nu_1 \geq \nu_2$ (b) $\nu_1 \perp \nu_2$ (c)$\nu_1 \to \nu_2$ (d) none of these	2	2
13.	A is a positive set with respect to the signed measure ν on $[\![X, \mathcal{S}]\!]$, if (a) $A \in \mathcal{S}$ and $\nu(E) \geq 0$ for all $E \subseteq A$ (b) $A \in \mathcal{S}$ and $\nu(E) > 0$ for all $E \subseteq A$ (c) $A \in \mathcal{S}$ and $\nu(E) \leq 0$ for all $E \subseteq A$ (d) $A \in \mathcal{S}$ and $\nu(E) = 0$ for all $E \subseteq A$	2	2

14.	The variance of Bernoulli distribution is (a) $np(1-p)$ (b) $p(1-p)$ (c) $\frac{1-p}{p}$ (d) 0	2	2
15.	Let X and Y be random variables on $(\Omega, \mathcal{F}, P)$ such that $E\|X\|^p < \infty$, $E\|Y\|^p < \infty$ for some $1 \leq p \leq \infty$. Then $(E\|X+Y\|^p)^{1/p} \leq (E\|X\|^p)^{1/p} + (E\|Y\|^p)^{1/p}$ (a) Cauchy-Schwarz inequality (b) Holder's inequality (c) Minkowski's inequality (d) Jensen's inequality	2	2

Q. No.	**SECTION C** $(\mathbf{2 \times 15 = 30})$ **Answer ANY TWO questions**	**CO**	**KL**
16.	Show that the outer measure of an interval equals its length.	3	3
17.	State and prove Fatou's lemma.	3	3
18.	Let f and g be integrable functions. Show that $af + bg$ is integrable and $\int (af + bg)\, d\mu = a \int f\, d\mu + b \int g\, d\mu$. If $f = g$ a.e. then $\int f\, d\mu = \int g\, d\mu$.	3	3
19.	Let $[\![X, \mathcal{S}, \mu]\!]$ and $[\![Y, \mathcal{J}, \nu]\!]$ be σ-finite measure spaces. For $V \in \mathcal{S} \times \mathcal{J}$ write $\phi(x) = \nu(V_x), \psi(y) = \mu(V^y)$, for each $x \in X, y \in Y$. Prove that ϕ is $\mathcal{S}$-measurable, ψ is $\mathcal{J}$-measurable and $\int_X \phi\, d\mu = \int_Y \psi\, d\nu$.	3	3

Q. No.	**SECTION D** $(\mathbf{2 \times 15 = 30})$ **Answer ANY TWO questions**	**CO**	**KL**
20.	Show that the following conditions on the set E are equivalent: (i) E is measurable (ii) for all $\epsilon > 0$ there exists an open set, $O \supseteq E$ such that $m^*(O - E) \leq \epsilon$, (iii) there exists a G_δ-set, G containing E such that $m^*(G - E) = 0$.	4	4
21.	Discuss Riemann integrable over $[a, b]$. Also prove that if f is Riemann integrable and bounded over the finite interval $[a, b]$, then f is integrable and $R\int_a^b f\, dx = \int_a^b f\, dx$.	4	4
22.	If μ is a measure on a σ- ring $\mathcal{S}$, show that $\overline{\mathcal{S}}$ of sets of the form $E \Delta N$ for any sets E, N such that $E \in \mathcal{S}$ where N is contained in some set in $\mathcal{S}$ of zero measure, is a σ-ring, and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\overline{\mathcal{S}}$.	4	4
23.	State and prove Lebesgue decomposition theorem.	4	4

Q. No.	**SECTION E** $(\mathbf{2 \times 10 = 20})$ **Answer ANY TWO questions**	**CO**	**KL**
24.	Prove that every interval is measurable.	5	5
25.	Construct a non-measurable set of [0,1].	5	5
26.	Let ν be a signed measure on $[\![X, \mathcal{S}]\!]$. Let $E \in \mathcal{S}$ and $\nu(E) > 0$. Prove that there exists A, a set positive with respect to ν, such that $A \subseteq E$ and $\nu(A) > 0$.	5	5
27.	If $\mathcal{A}$ is an algebra, prove that $\mathcal{S}(\mathcal{A}) = \mathcal{M}_0(\mathcal{A})$.	5	5

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