STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2023 – 24)

M. Sc. DEGREE EXAMINATION, NOVEMBER 2023
BRANCH I - MATHEMATICS
FIRST SEMESTER

COURSE : CORE
PAPER : ABSTRACT ALGEBRA
SUBJECT CODE : 23MT/PC/AA14
TIME : 3 HOURS **MAX. MARKS : 100**

Q. No.	SECTION A ($5 \times 2 = 10$) Answer ALL questions	CO	KL
1.	Define conjugacy relation on a group G and what is the number of conjugacy classes of the multiplicative group $G = \{1, -1, i, -i\}$.	1	1
2.	Define a unit in a ring and write down all the units of the ring Z of integers.	1	1
3.	Define the content of a polynomial with integer coefficients.	1	1
4.	Define a finite extension and an algebraic extension of a field.	1	1
5.	Define a normal extension of a field and give an example.	1	1

Q. No.	SECTION B ($10 \times 1 = 10$) Answer ALL questions	CO	KL
6.	Which of the following is/are true (a). The number of conjugacy classes of the symmetric group S_4 is 5. (b). There is no non-abelian group of order 169. (c). Any group G of even order contains an element $a \neq e$ in G such that $a^{-1} = a$ (d). Any group G of order 625 contains an element $a \neq e$ such that $ax = xa$, for every $x \in G$	2	2
7.	Which of the following is/are true (a). There is a subgroup of order 8 in a group of order 56. (b). The number of non-isomorphic abelian groups of order 625 is 4. (c). There is a unique abelian group of order 31. (d). Any two subgroups of order 4 in a group of order 36 are conjugates.	2	2

8.	Which of the following statement(s) is/are false. (a). $3 + 4i$ is a unit in the ring Gaussian integers (b). The units of a field F are the non-zero elements of F. (c). Any field is an Euclidean ring. (d). The ring of integers is a unique factorization domain.	2	2
9.	Which of the following statement(s) is/are false (a). The numbers 4 and -4 are associates in the ring Z of integers. (b). The ideal generated by a prime element of an Euclidean ring R is a maximal ideal of R. (c). There are Euclidean rings without multiplicative identity element. (d). The ring Z of integers is a principal ideal ring.	2	2
10.	Which of the following statement(s) is/are true. (a). There is an irreducible polynomial of degree 3 over the field of real numbers. (b). The ideal in the ring $R[X]$ of polynomials in X over the field R of real numbers generated by the polynomial $X^2 + 1$ is a maximal ideal of $R[X]$. (c). The polynomial $X^2 - 2$ is irreducible over the field R of real numbers. (d). The ring $Z[X]$, of polynomials in X over the ring Z of integers, is an integral domain	2	2
11.	Which of the following statement(s) is/are true. (a). The polynomial ring $F[X]$ over the field F is a principal ideal ring. (b). The polynomial ring $J_n[X]$ in the variable X over the ring J_n of residue classes of integers modulo n is an integral domain. (c). A product of two primitive polynomials is primitive. (d). The polynomial $X^2 + X + 1$ is irreducible over the field of real numbers.	2	2
12.	Which of the following statement(s) is/are true (a). There is no field F such that $R \subset F \subset C$, where R is the field of real numbers and C is the field of complex numbers. (b). Any finite extension of a field F is an algebraic extension of F. (c). Any algebraic extension of a field F is a finite extension of F. (d). The field R of real numbers is a finite extension of the field Q of rational numbers.	2	2

13.	Which of the following statement(s) is/are true (a). Any irreducible polynomial over the field R of real numbers has distinct roots. (b). An irreducible polynomial over the field Q of rational numbers has multiple roots. (c). Any simple extension of a field F is a finite extension of F. (d). The splitting field of the polynomial $X^2 - 2$ over the field Q of rational numbers is the field R of real numbers.	2	2
14.	Which of the following statement(s) is/are false (a). The field of complex numbers is a normal extension of the field of real numbers. (b). The symmetric group S_6 is solvable. (c). Any homomorphic image of a solvable group is solvable. (d). The field $Q(\sqrt{2})$ is a normal extension of the field Q of rational numbers.	2	2
15.	Which of the following statement(s) is/are false? (a). The splitting field of a polynomial over a field F is always a normal extension of F. (b). If C is the field of complex numbers and R is the field of real numbers, then $G(C, R)$ is a group of order 2. (c). If C is the field of complex numbers and R is the field of real numbers, then the fixed field of $G(C, R)$ is C. (d). The set of automorphisms of a field K is a group under composition of functions.	2	2

Q. No.	**SECTION C ($2 \times 15 = 30$)** **Answer ANY TWO questions**	**CO**	**KL**
16.	Prove that any finite abelian group is a direct product of cyclic groups.	3	3
17.	Prove that the ring of Gaussian integers is a Euclidean ring.	3	3
18.	(a). State and prove Eisenstein's criterion for irreducibility of polynomial over the field of rational numbers. (b). For a prime p, prove that the polynomial $X^{p-1} + X^{p-2} + \dots + X + 1$ is irreducible over the field of rational numbers. (10+5)	3	3

19.	(a). If a rational number *r* is an algebraic integer, prove that *r* must be an ordinary integer. (b). If *a*, *b* in *K* are algebraic over *F* of degrees *m* and *n*, respectively, and if *m* and *n* are relatively prime, prove that *F(a, b)* is of degree *mn* over *F*. (5+10)	3	3

Q. No.	**SECTION D** $(\mathbf{2 \times 15 = 30})$ **Answer ANY TWO questions**	CO	KL
20.	State and prove the fundamental theorem of Galois theory	4	4
21.	(a). If $p(x)$ is a polynomial in $F[x]$ of degree $n \geq 1$ and is irreducible over *F*, then prove that there is an extension *E* of *F*, such that $[E:F] = n$, in which $p(x)$ has a root. (b). Find the splitting field of the polynomial $X^3 - 2$ over the field of rational numbers. Also find the degree of the splitting field over the field Q rational numbers. (8+7)	4	4
22.	Prove that any group of order $11^2 . 13^2$ is abelian	4	4
23.	Prove that two abelian groups of order p^n are isomorphic if and only if they have the same invariants.	4	4

Q. No.	**SECTION E** $(\mathbf{2 \times 10 = 20})$ **Answer ANY TWO questions**	CO	KL
24.	Find the conjugacy classes of the Symmetric group S_3. Also find the normalizer of each element of the symmetric group S_3. Further verify the class equation for the symmetric group S_3	5	5
25.	Prove that a necessary and sufficient condition that the element *a* in the Euclidean ring be a unit is that $d(a) = d(1)$.	5	5
26.	Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.	5	5
27.	Find the group $G(K, Q)$, where Q is the field of rational numbers and $K = Q(\sqrt[3]{2})$, where $\sqrt[3]{2}$ is the real cube root of 2. Also find the fixed field of $G(K, Q)$.	5	5