STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2019-20 & thereafter)

SUBJECT CODE : 19MT/PC/LA24

M. Sc. DEGREE EXAMINATION, APRIL 2023
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : LINEAR ALGEBRA
TIME : 3 HOURS **MAX. MARKS : 100**

Section-A
Answer ALL the questions (5x2=10)

1. Define similar matrices and what is the number of similarity classes of 4 x 4 nilpotent matrices over a field *F*?
2. Define the Companion matrix of the polynomial $f(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_{r-1} x^{r-1} + x^r \in F[x]$.
3. If $T : R^2 \to R^2$ is a linear transformation whose matrix with respect to the standard basis of R^2 is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then find all the subspaces of R^2 invariant under *T*.
4. Define an orthogonal matrix and what can you say about the determinant of an orthogonal matrix?
5. Define a bilinear form on a vector space and give an example.

Section-B
Answer any FIVE questions (5x6=30)

6. If *S* and *T* are nilpotent transformations which commute, prove that *ST* and *S* + *T* are nilpotent transformations.
7. Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of *V* invariant under *T*. Let T_1 and T_2 be the linear transformation induced by *T*on V_1 and V_2, respectively. If the minimal polynomial of T_1 over *F* is $p_1(x)$ while that of T_2 is $p_2(x)$, then prove that the minimal polynomial for *T* over *F* is the least common multiple of $p_1(x)$ and $p_2(x)$.
8. Let *T* be a linear operator on an n-dimensional vector space *V*. Then prove that the characteristic and minimal polynomials for *T* have the same roots, except for multiplicities.
9. Prove that for any linear operator *T* on a finite dimensional inner product space *V*, there is a unique linear operator T^* on *V* such that $(T\alpha \mid \beta) = (\alpha \mid T^*\beta)$, for all $\alpha, \beta \in V$.
10. Let *V* be a complex vector space and *f* a form on *V* such that $f(\alpha, \alpha)$ is real for every α. Then prove that *f* is Hermitian.
11. If *V* is *n*-dimensional vector space over *F* and if $T \in A(V)$ has all its characteristic roots in *F*, then prove that *T* satisfies a polynomial of degree n over *F*.
12. Let *F* be the field of real numbers or the field of complex numbers. Let *A* be an $n \times n$ matrix over *F*. The function *g* defined by $g(X,Y) = Y^* A X$ is a positive form on the space $F^{n \times 1}$ if and only if there is an invertible $n \times n$ matrix *P* with entries in *F* such that $A = P * P$.

Section-C

Answer any THREE questions **(3x20=60)**

13. (a) If $T \in A_F(V)$ is nilpotent, of index of nilpotent n_1, then prove that there is a basis of V in which the matrix of T has the form $\begin{bmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_r} \end{bmatrix}$ where $n_1 \geq n_2 \geq \cdots \geq n_r$ and where $n_1 + n_2 + \cdots + n_r = \dim_F(V)$.

(b) Find the characteristic roots of the matrix $\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}$. Is this triangulizable over the field of rational numbers? Give reasons for your answer. (15+5)

14. (a) Prove that two elements S and T in $A_F(V)$ are similar in $A_F(V)$ if and only if they have the same elementary divisors.

(b) Suppose the two matrices A, B in F_n are similar in K_n where K is an extension of F. Prove that A and B are already similar in F_n. (15+5)

15. (a) Let T be a linear operator on a finite dimensional vector space V. If f is the characteristic polynomial for T, then prove that $f(T) = 0$.

(b) Let V be a finite dimensional vector space over a field F and let T be a linear operator on V. Then prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p(x) = (x - c_1)(x - c_2)\cdots(x - c_k)$, where $c_1, c_2, , \cdots, c_k$ are distinct elements. (10+10)

16. (a) Let V be a finite-dimensional inner product space, and f a linear functional on V. Then prove that there exists a unique vector β in V such that $f(\alpha) = (\alpha|\beta)$, for all α in V.

(b) Whether the result given in (a) is true if V is infinite dimensional? If it is true, prove the result for infinite dimensional inner product space. If the result is false, give a counter example to disprove it. (10+10)

17. (a) Let V be a finite dimensional inner product space and f be form on V. Prove that there is a unique linear operator T on V such that $f(\alpha, \beta) = (T\alpha \mid \beta)$, for all α, β in V, and the map $f \to T$ is an isomorphism of space of forms onto $L(V, V)$.

(b) State and prove Principal Axis theorem. (10+10)