STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2019-20 & thereafter)

SUBJECT CODE : 19MT/MC/VL64

B. Sc. DEGREE EXAMINATION, APRIL 2023
BRANCH I – MATHEMATICS
SIXTH SEMESTER

COURSE : MAJOR CORE
PAPER : VECTOR SPACES AND LINEAR TRANSFORMATIONS
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

ANSWER ANY TEN QUESTIONS. **(10 × 2 = 20)**

1. Define vector space.
2. Let $\boldsymbol{W}$ be the set of vectors of the form $(\boldsymbol{a}, \boldsymbol{a}, \boldsymbol{a} + \mathbf{2})$. Show that $\boldsymbol{W}$ is not a subspace of $\boldsymbol{R}^{\mathbf{3}}$.
3. Define rank of a matrix.
4. State orthogonal matrix theorem.
5. Define an orthogonal matrix.
6. Find the co-ordinate vector of $\boldsymbol{v} = (\mathbf{4}, \mathbf{0}, -\mathbf{2})$ in $\boldsymbol{R}^{\mathbf{3}}$ relative to the orthonormal basis $\boldsymbol{B} = \{(\mathbf{1}, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{0}, \mathbf{1})\}$.
7. Define Isomorphic vector spaces.
8. Prove that the functions $(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{2}}$ and $\boldsymbol{g}(\boldsymbol{x}) = \mathbf{4}\boldsymbol{x} - \mathbf{3}$ over [0, 1] are orthogonal.
9. If $\boldsymbol{u} = (\mathbf{2} + \mathbf{3}\boldsymbol{i}, -\mathbf{1} + \mathbf{5}\boldsymbol{i})$, $\boldsymbol{v} = (\mathbf{1} + \boldsymbol{i}, -\boldsymbol{i})$ in $\boldsymbol{C}^{\mathbf{2}}$. Compute $\|\boldsymbol{u}\|$ & $\|\boldsymbol{v}\|$ & $(\boldsymbol{u}, \boldsymbol{v})$.
10. If $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x}$, $\boldsymbol{g}(\boldsymbol{x}) = \mathbf{4} - \mathbf{2}\boldsymbol{x}$. Compute $\boldsymbol{f} + \boldsymbol{g}$, $\mathbf{3}\boldsymbol{f}$ and $-\boldsymbol{g}$.
11. Define matrix transformation.
12. When a matrix is said to be orthogonally diagonalizable?

SECTION –B

ANSWER ANY FIVE QUESTIONS. **(5 × 8 = 40)**

13. Let V be a vector space, $\mathbf{0}$ the zero vector of $\boldsymbol{V}$, c a scalar. Then prove that
    (a) $\mathbf{0}\boldsymbol{v} = \mathbf{0}$ (b) $c\mathbf{0} = \mathbf{0}$ (c) $(-\mathbf{1}) = \boldsymbol{v}$
    (d) If $\boldsymbol{cv} = \mathbf{0}$ then either $\boldsymbol{c} = \mathbf{0}$ $\boldsymbol{or}$ $\boldsymbol{v} =$ .0
14. Find a basis for the row space of the following matrix A and determine its rank
    $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$
15. Prove that an orthogonal set of non-zero vectors in a vector space is linearly independent.
16. Let $\boldsymbol{B}$ and $\boldsymbol{B}'$ be bases for a vector space $\boldsymbol{U}$ and $\boldsymbol{P}$ be the transition matrix from $\boldsymbol{B}$ to $\boldsymbol{B}'$. Then prove $\boldsymbol{P}$ is invertible and the transition matrix from $\boldsymbol{B}'$ to $\boldsymbol{B}$ is $\boldsymbol{P}^{-\mathbf{1}}$.
17. Find the fourth order Fourier approximation to $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$ over the interval $[-\boldsymbol{\pi}, \boldsymbol{\pi}]$.
18. Prove that the set $\{(\mathbf{1}, \mathbf{3}, -\mathbf{1}), (\mathbf{2}, \mathbf{1}, \mathbf{0}), (\mathbf{4}, \mathbf{2}, \mathbf{1})\}$ is a basis for $\boldsymbol{R}^{\mathbf{3}}$.
19. Prove that a linear transformation $\boldsymbol{T}$ is one-to-one if and only if the kernel is the zero vector.

SECTION –C

ANSWER ANY TWO QUESTIONS. **(2 × 20 = 40)**

20. (a) Prove that the set $\boldsymbol{U}$ of $\mathbf{2} \times \mathbf{2}$ diagonal matrices is a subspace of the vector spaces $\boldsymbol{M_{22}}$ of $\mathbf{2} \times \mathbf{2}$ matrices.

    (b) Determine whether the vector $(\mathbf{8}, \mathbf{0}, \mathbf{5})$ is a linear combination of the vectors $(\mathbf{1}, \mathbf{2}, \mathbf{3}), (\mathbf{0}, \mathbf{1}, \mathbf{4})$ and $(\mathbf{2}, -\mathbf{1}, \mathbf{1})$ in $\boldsymbol{R^3}$.

21. (a) State and prove Gram Schmidt Orthogonalization process.

    (b) Solve the following homogenous system of linear equations. Interpret the setof solutions as a subspace, sketch the subspace of solutions.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -x_1 + x_2 \qquad &= 0 \\ x_1 + x_2 + 4x_3 &= 0. \end{aligned}$$

22. (a) (i) Show that the matrix *A* is diagonalizable and (ii) Find a diagonal matrix *D* similar to *A* where A = $\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$.

    (b) Find the least squares line for the following data points $(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2.4}), (\mathbf{3}, \mathbf{3.6}), (\mathbf{4}, \mathbf{4})$.

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