STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted during the academic year 2019 – 20 and thereafter)

SUBJECT CODE : 19MT/PC/PS34

M. Sc. DEGREE EXAMINATION, NOVEMBER 2022
BRANCH I - MATHEMATICS
THIRD SEMESTER

COURSE : CORE
PAPER : PROBABILITY AND STOCHASTIC PROCESSES
TIME : 3 HOURS **MAX. MARKS : 100**

SECTION – A

ANSWER ALL THE QUESTIONS: (5 × 2 = 10)

1. State Borel-Cantelli Lemma.
2. Define stochastic process.
3. If i is recurrent and $i \leftrightarrow j$, then j is recurrent.
4. Define continuous – time Markov chain.
5. Define Submartingale and Supermartingale.

SECTION – B

ANSWER ANY FIVE QUESTIONS: (5 × 6 = 30)

6. At a party n people put their hats in the centre of a room where the hats are mixed together. Each person then randomly selects one. Find mean and variance of X, the number that select their own hat.

7. Let $\tau_1, \tau_2, \ldots, \tau_n$ denote the ordered values from a set of n independent uniform $(0, t)$ random variable. Let $Y_1, Y_2, \ldots$ be independent and identically distributed nonnegative random variables that are also independent of $\{\tau_1, \tau_2, \ldots, \tau_n\}$. Then
$P\{Y_1 + \cdots + Y_k < \tau_k, k = 1, \ldots, n | Y_1 + \cdots + Y_n = y\}$
$= \begin{cases} 1 - y/t & 0 < y < t \\ 0 & otherwise \end{cases}$.

8. State and prove Limiting probabilities for the Embedded $M/G/1$ Queue.

9. Consider a Yule process with $X(0) = 1$. Let the sum of the ages at time t, $A(t)$, can be expressed as $A(t) = a_0 + t + \sum_{i=1}^{X(t)-1}(t - S_i)$, where $a_0$is the age at $t = 0$ of the initial individual. Compute $E[A(t)]$ condition on $X(t)$.

10. State and prove the Martingale Convergence theorem.

11. In an election, candidate A receives n votes and candidate B receives m votes, where $n > m$. Assuming that all orderings are equally likely, show that the probability that A is always ahead in the count of votes in $(n-m)/(n+m)$.

12. State and prove Kolmogorov's Backward equations.

SECTION – C

ANSWER ANY THREE QUESTIONS: **(3 × 20 = 60)**

13. a) If $\{E_n, n \geq 1\}$ is either an increasing or decreasing sequence of events,

then $\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right)$.

b) Prove that $Var(Y) = E[N]Var(X) + E^2[X]Var(N)$. (10 + 10)

14. Derive the differential equation $P_n(t) \; = \; e^{-\lambda t}\frac{(\lambda t)^n}{n!}$.

15. a) Consider a gambler who at each play of the game has probability p of winning 1 unit and probability $q = 1 - p$ of losing 1 unit. Assuming successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0.

b) Prove that

(i) if $p \geq 1/n$, then $\overline{\prod_i} \leq \prod_i$ for all i.

(ii) if $p \leq 1/n$, then $\overline{\prod_i} \geq \prod_i$ for all i. (15 + 5)

16. Determine the limiting probabilities for a birth and death process.

17. State and prove Azuma's Inequality for Martingales.