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Ramanujan's modular equations of degree 5

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Abstract. We provide alternative derivations of theta function identities associated with modular equations of degree 5. We then use the identities to derive the corresponding modular equations.

Keywords. Theta-function; elliptic integral; modular equation; multiplier.

1. Introduction

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \tag{1.1}$$

where $|ab| < 1$. If we set $a = q\mathrm{e}^{2iz}$, $b = q\mathrm{e}^{-2iz}$, and $q = \mathrm{e}^{\pi i \tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation (p. 464 of [21]).

Throughout the paper it is assumed that $|q| < 1$. Also, as usual, for any complex number a, we define

$$(a;q)_0 = 1, \quad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}) \quad \text{for } n \geq 1 \quad \text{and}$$
$$(a;q)_\infty := \prod_{k=1}^{\infty}(1-aq^{k-1}). \tag{1.2}$$

Jacobi's famous triple product identity can be written as (p. 35, Entry 19 of [5])

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.3}$$

Three special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;\ q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.4}$$

$$\psi(q) := f(q, q^3) = \frac{1}{2} f(1, q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.5}$$

$$\begin{aligned} f(-q) := f(-q, -q^2) &= \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2} \\ &= (q; q)_\infty. \end{aligned} \tag{1.6}$$

If we write $q = e^{2\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$, then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the classical Dedekind eta-function.

Next, we give the definition of a modular equation as understood by Ramanujan. The complete elliptic integral of the first kind $K(k)$ is defined as

$$\begin{aligned} K(k) &:= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} \\ &= \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad (0 < k < 1) \end{aligned} \tag{1.7}$$

where the series representation is found by expanding the integrand in a binomial series and integrating termwise. The number k is called the modulus of K, and $k' := \sqrt{1 - k^2}$ is called the complementary modulus. Let K, K', L and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n \frac{K'}{K} = \frac{L'}{L} \tag{1.8}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.8). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The corresponding multiplier m is defined by

$$m = \frac{K}{L}. \tag{1.9}$$

Ramanujan recorded several modular equations of degree 5 and associated theta-function identities in his notebooks [17] and the lost notebook (pp. 50 and 56 of [18]). As Ramanujan did not provide any proofs for his results, one can only speculate his proofs. Berndt (see Chapter 19 of [5], p. 202, Entry 50 of [6] and pp. 363–367 of [7]), Shen [19] and Kang [15] proved Ramanujan's theta-function identities. Baruah and Bhattacharyya [4] also found alternative proofs for some of the theta-function identities. Berndt (Entry 13, pp. 280–282 of [5]) also proved all of Ramanujan's modular equations of degree 5. In most of the proofs, Berndt used a method of parametrizations, which require prior knowledge of the equations. But, provably Ramanujan first derived a theta-function identity and then transcribed it into an equivalent modular equation by using his catalogue of theta-functions (pp. 122–124, Entries 10–12 of [5]). Berndt sometimes reversed this process to derive Ramanujan's theta-function identities. A lot more is now known about Ramanujan's work, now that Berndt has finished editing Ramanujan's notebooks. It may be worthwhile, with the hindsight, to revisit Berndt's proofs and see if they can be improved. In this paper, we find alternative proofs of some of the associated theta-function identities and then transcribe these and their different combinations to arrive at Ramanujan's modular equations of degree 5.

We organize our paper as follows. In §2, we state some preliminary results including the theta function identities for which simpler and direct proofs are available elsewhere. In §3, we state some theta function identities which we think were fundamental in Ramanujan's method. We also provide alternative proofs of some other theta function identities. In the final section, we prove the modular equations by using results from the previous two sections.

2. Preliminary results

In this section, we state some results which will be used to derive our theta-function identities in the next section.

Lemma 2.1 (*p*. 39, *Entry* 24 *of* [5]). *We have*

$$\frac{\psi(q)}{\psi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}}, \tag{2.1}$$

$$f^3(-q) = \varphi^2(-q)\psi(q), \tag{2.2}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}, \tag{2.3}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q), \qquad \chi(q)\chi(-q) = \chi(-q^2), \tag{2.4}$$

where $\chi(q)$ *is defined by*

$$\chi(q) := (-q; q^2)_\infty = \prod_{k=0}^{\infty}(1 + q^{2k+1}). \tag{2.5}$$

Lemma 2.2 (*p*. 40, *Entry* 25 *of* [5]). *We have*

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{2.6}$$

$$\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2), \tag{2.7}$$

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{2.8}$$

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{2.9}$$

Lemma 2.3 (*p*. 276, *Lemma of* [5]). *We have*

$$f(q)f(-q) = f(-q^2)\varphi(-q^2). \tag{2.10}$$

Lemma 2.4 (*p*. 45, *Entry* 29 *of* [5]). *If* $ab = cd$, *then*

$$f(a, b)f(c, d) + f(-a, -b)f(-c, -d) = 2f(ac, bd)f(ad, bc) \tag{2.11}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{2.12}$$

Lemma 2.5 (*pp.* 122–124, *Entries* 10–12 *of* [5]). *If*

$$z = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \quad \text{and} \quad y = \pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)},$$

then

$$\varphi(e^{-y}) = \sqrt{z}, \tag{2.13}$$

$$\varphi(-e^{-y}) = \sqrt{z}(1-x)^{1/4}, \tag{2.14}$$

$$\varphi(-e^{-2y}) = \sqrt{z}(1-x)^{1/8}, \tag{2.15}$$

$$\varphi(e^{-y/4}) = \sqrt{z}(1+x^{1/4}), \tag{2.16}$$

$$\varphi(-e^{-y/4}) = \sqrt{z}(1-x^{1/4}), \tag{2.17}$$

$$\psi(e^{-y}) = \sqrt{z/2}(xe^{y})^{1/8}, \tag{2.18}$$

$$\psi(-e^{-y}) = \sqrt{z/2}\{x(1-x)e^{y}\}^{1/8}, \tag{2.19}$$

$$\psi(e^{-2y}) = \frac{1}{2}\sqrt{z}(xe^{y})^{1/4}, \tag{2.20}$$

$$f(-e^{-2y}) = \sqrt{z}2^{-1/3}\{x(1-x)e^{y}\}^{1/12}, \tag{2.21}$$

$$f(e^{-y}) = \sqrt{z}2^{-1/6}\{x(1-x)e^{y}\}^{1/24}, \tag{2.22}$$

$$f(-e^{-4y}) = \sqrt{z}4^{-1/3}(1-x)^{1/24}(xe^{y})^{1/6}, \tag{2.23}$$

$$f(-e^{-y}) = \sqrt{z}2^{-1/6}(1-x)^{1/24}(xe^{y})^{1/24}, \tag{2.24}$$

$$\chi(e^{-y}) = 2^{1/6}\{x(1-x)e^{y}\}^{-1/24}, \tag{2.25}$$

$$\chi(-e^{-y}) = 2^{1/6}(1-x)^{1/12}(xe^{y})^{-1/24}. \tag{2.26}$$

3. Theta-function identities

In this section, we state and prove some theta-function identities. We think that the proofs of those identities recorded by Ramanujan and presented here are more transparent than those found by Berndt [5–7]. We also mention that the identities in [7] were likely unknown to the author when [5] was written.

We observe that the following four theta function identities were central in Ramanujan's method of deriving the modular equations of degree 5.

Theorem 3.1. *We have*

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7) = 4q\chi(q)f(-q^5)f(-q^{20}), \tag{3.1}$$

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3) = \frac{\varphi(-q^5)f(-q^5)}{\chi(-q)}, \tag{3.2}$$

$$5\varphi^2(q^5) - \varphi^2(q) = 4\chi(q)\chi(q^5)\psi^2(-q), \tag{3.3}$$

$$\psi^2(q) - 5q\psi^2(q^5) = \frac{\varphi^2(-q)}{\chi(-q)\chi(-q^5)}. \tag{3.4}$$

The later equalities in (3.1) and (3.2) were recorded by Ramanujan in Entry 9(vii) of Chapter 19 of his second notebook [17] and these readily follow from the Jacobi triple product identity (1.3) (see Berndt's proof in pp. 259–261 of [5]). The identities (3.1) and (3.2) were recorded by Ramanujan in Entry 9(iii) and Entry 10(v) of the same chapter of the second notebook. Raghavan and Rangachari [16] proved (3.2) and (3.4) by using the theory of Hauptmoduls. Berndt (p. 258 and p. 263 of [5]) and Shen [19] proved these by using Lambert series and elliptic functions. A more constructive and straightforward derivation was found by Son [20]. The identities (3.2) and (3.4), with q replaced by $-q$, were also recorded by Ramanujan in his first notebook (p. 295, Vol. I of [17]) (see also Berndt's book pp. 365–366, Entries 18–19 of [7]). Incidently, (3.2) and (3.4) can be derived from (3.1) and (3.3), respectively, by using modular transformations. Furthermore, we see that the last two identities can be derived from the first two with the aid of the elementary identities in Lemmas 2.1 and 2.2. In Theorem 2.1 of [4], Baruah and Bhattacharyya derived (3.4) from (3.1) and (3.2). In the following we establish (3.3) from (3.1) and (3.2).

Firstly, with the help of (2.3), we rewrite (3.1) in the form

$$\varphi^2(q) - \varphi^2(q^5) = 4q\chi(q)\chi(q^5)\psi^2(-q^5). \tag{3.5}$$

Next, replacing q by $-q$ in (3.2), we have

$$q\psi^2(-q^5) = \frac{\varphi(q^5)f(q^5)}{\chi(q)} - \psi^2(-q). \tag{3.6}$$

Employing (3.6) in (3.5), and also using (2.3), we find that

$$\begin{aligned}\varphi^2(q) - \varphi^2(q^5) &= 4\chi(q)\chi(q^5)\left\{\frac{\varphi(q^5)f(q^5)}{\chi(q)} - \psi^2(-q)\right\}\\ &= 4\chi(q^5)\varphi(q^5)f(q^5) - 4\chi(q)\chi(q^5)\psi^2(-q)\\ &= 4\varphi^2(q^5) - 4\chi(q)\chi(q^5)\psi^2(-q),\end{aligned} \tag{3.7}$$

which is clearly equivalent to (3.3).

Now we present a proof of (3.1) which is only slightly different from that of Son [20]. Setting $a = q, b = -q^4, c = -q^2$ and $d = q^3$ in (2.11) and (2.12), we find that

$$f(q,-q^4)f(-q^2,q^3) + f(-q,q^4)f(q^2,-q^3) = 2f(q^4,q^6)f(-q^3,-q^7) \tag{3.8}$$

and

$$f(q,-q^4)f(-q^2,q^3)-f(-q,q^4)f(q^2,-q^3)=2qf(q^2,q^8)f(-q,-q^9). \tag{3.9}$$

Now, by Jacobi triple product identity (1.3) and the product representation (1.6) for $f(-q)$, we have

$$f(-q,-q^4)f(-q^2,-q^3)=f(-q)f(-q^5), \tag{3.10}$$

which was one of the three identities of Entry 9(vii) recorded by Ramanujan in Chapter 19 of his second notebook (p. 258 of [5]), the other two being the last equalities of (3.1) and (3.2) as mentioned earlier.

Employing (3.10), with q replaced by $-q$, and the last equality of (3.2) in (3.8) and (3.9), we find that

$$f(q)f(q^5)+\frac{\varphi(q^5)f(q^5)}{\chi(q)}=2f(q^4,q^6)f(-q^3,-q^7) \tag{3.11}$$

and

$$f(q)f(q^5)-\frac{\varphi(q^5)f(q^5)}{\chi(q)}=2qf(q^2,q^8)f(-q,-q^9). \tag{3.12}$$

With the help of (2.3) we can rewrite the above identities as

$$\varphi(q)+\varphi(q^5)=2f(q^4,q^6)f(-q^3,-q^7)\frac{\chi(q)}{f(q^5)} \tag{3.13}$$

and

$$\varphi(q)-\varphi(q^5)=2qf(q^2,q^8)f(-q,-q^9)\frac{\chi(q)}{f(q^5)}. \tag{3.14}$$

Now, multiplying (3.13) and (3.14) and using the second equalities of (3.1) and (3.2), and then with the help of (2.3) and (2.4), we deduce that

$$\begin{aligned}\varphi^2(q)-\varphi^2(q^5)&=4qf(q^2,q^8)f(q^4,q^6)f(-q,-q^9)f(-q^3,-q^7)\frac{\chi^2(q)}{f^2(q^5)}\\&=4q\frac{\varphi(-q^{10})f(-q^{10})}{\chi(-q^2)}\cdot\chi(-q)f(q^5)f(q^{20})\cdot\frac{\chi^2(q)}{f^2(q^5)}\\&=4q\chi(q)f(-q^5)f(q^{20}).\end{aligned} \tag{3.15}$$

Thus we complete the proof of (3.1).

Remark 3.2. Further comments regarding Theorem 3.1 have been given in [11]. There, it was shown that any two of (3.1)–(3.4) can be deduced from the other two by elementary means. Moreover, the identities (3.1)–(3.4) can be stated in terms of eta-functions to show the symmetries among the identities. Then, any identity can be deduced from any other one by an Atkin–Lehner involution.

The refinements (3.13) and (3.14) of (3.1) and analogous results involving the function ψ were also recorded by Ramanujan in his lost notebook (p. 56 of [18]) in the forms

$$\varphi(q)+\varphi(q^5)=2q^{4/5}f(q,q^9)R^{-1}(q^4), \tag{3.16}$$

$$\varphi(q)-\varphi(q^5)=2q^{1/5}f(q^3,q^7)R(q^4), \tag{3.17}$$

$$\psi(q^2)+q\psi(q^{10})=q^{1/5}f(q^2,q^8)R^{-1}(q), \tag{3.18}$$

$$\psi(q^2)-q\psi(q^{10})=q^{-1/5}f(q^4,q^6)R(q), \tag{3.19}$$

where $R(q)$ denotes the famous Rogers–Ramanujan continued fraction defined by

$$R(q):=\frac{q^{1/5}}{1}+\frac{q}{1}+\frac{q^2}{1}+\frac{q^3}{1}+\cdots,\quad |q|<1.$$

Proofs can be found in [10,15,12,13].

The identities in the next two theorems were recorded by Ramanujan in his lost notebook (p. 50 and p. 56 of [18], and first proved by Kang [15]. Incidentally, all these identities now follow immediately from (3.1)–(3.4) with applications of Lemma 2.1.

Theorem 3.3 (p. 56 of [18], p. 93, Theorem 2.1 of [15]). *We have*

$$\frac{f^3(-q)}{f^3(-q^5)}=\frac{\psi(q)}{\psi(q^5)}\times\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)},$$

$$\frac{f^6(-q^2)}{f^6(-q^{10})}=\frac{\psi^4(q)}{\psi^4(q^5)}\times\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)},$$

$$\frac{f^3(-q^2)}{qf^3(-q^{10})}=\frac{\varphi(q)}{\varphi(q^5)}\times\frac{5\varphi^2(q^5)-\varphi^2(q)}{\varphi^2(q)-\varphi^2(q^5)},$$

$$\frac{f^6(-q)}{qf^6(-q^5)}=\frac{\varphi^4(-q)}{\varphi^4(-q^5)}\times\frac{5\varphi^2(-q^5)-\varphi^2(-q)}{\varphi^2(-q^5)-\varphi^2(-q)}.$$

Theorem 3.4 (p. 56 of [18], p. 95, Corollary 2.3 of [15]). *We have*

$$\big(\varphi^2(q)-\varphi^2(q^5)\big)\big(5\varphi^2(q^5)-\varphi^2(q)\big)=16qf^2(-q^2)f^2(-q^{10}), \tag{3.20}$$

$$\big(\psi^2(q)-q\psi^2(q^5)\big)\big(\psi^2(q)-5q\psi^2(q^5)\big)=f^2(-q)f^2(-q^5). \tag{3.21}$$

Theorem 3.5 (p. 278 of [5]; Equation (2.3) of [1]). *We have*

$$\varphi(-q^5)\varphi(q)-\varphi(q^5)\varphi(-q)=4qf(-q^4)f(-q^{20}). \tag{3.22}$$

Proof. Setting $a=q$, $b=-q^4$, $c=-q^2$ and $d=q^3$ in (2.11), we find that

$$f(q,-q^4)f(-q^2,q^3)+f(-q,q^4)f(q^2,-q^3)=2f(-q^3,-q^7)f(q^4,q^6). \tag{3.23}$$

Employing (3.10) and the second equality of (3.2) in (3.23), we obtain

$$f(q)f(q^5)+\frac{\varphi(q^5)f(q^5)}{\chi(q)}=2f(-q^3,-q^7)f(q^4,q^6). \tag{3.24}$$

With the help of (2.3) we rewrite (3.24) as

$$\varphi(q)+\ \varphi(q^5)=2\frac{\chi(q)f(-q^3,-q^7)f(q^4,q^6)}{f(q^5)}. \tag{3.25}$$

Again, setting $a=-q$, $b=-q^4$, $c=-q^2$ and $d=-q^3$ in (2.12), we obtain

$$f(-q,-q^4)f(-q^2,-q^3)-f(q,q^4)f(q^2,q^3)=-2qf(q^2,q^8)f(q,q^9). \tag{3.26}$$

Employing (3.10) and the second equality of (3.2) in (3.23), we find that

$$f(-q)f(-q^5)-\ \frac{\varphi(-q^5)f(-q^5)}{\chi(-q)}=-2qf(q^2,q^8)f(q,q^9), \tag{3.27}$$

which can also be written, with the aid of (2.3), as

$$\varphi(-q^5)-\varphi(-q)=2q\frac{\chi(-q)f(q^2,q^8)f(q,q^9)}{f(-q^5)}. \tag{3.28}$$

Multiplying (3.25) and (3.28), and then employing (2.4), (2.6), (2.10), and the second equality of (3.2), we find that

$$\begin{aligned}&\varphi(q)\varphi(-q^5)-\varphi(-q)\varphi(q^5)-\varphi^2(-q^2)+\varphi^2(-q^{10})\\&\quad=4qf(q,q^9)f(-q^3,-q^7).\end{aligned} \tag{3.29}$$

Now, setting $a=q$, $b=q^9$, $c=-q^3$ and $d=-q^7$ in (2.11) and (2.12), we find that

$$\begin{aligned}&f(q,q^9)f(-q^3,-q^7)+\ f(-q,-q^9)f(q^3,q^7)\\&\quad=2f(-q^4,-q^{16})f(-q^8,-q^{12})\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}&f(q,q^9)f(-q^3,-q^7)-\ f(-q,-q^9)f(q^3,q^7)\\&\quad=2qf(-q^6,-q^{14})f(-q^2,-q^{18}).\end{aligned} \tag{3.31}$$

Adding (3.30) and (3.31) and then using (3.10) and the second equality of (3.1), we deduce that

$$f(q,q^9)f(-q^3,-q^7)=f(-q^4)f(-q^{20})+q\chi(-q^2)f(q^{10})f(-q^{40}). \tag{3.32}$$

Employing (3.32) in (3.29) and then with the aid of (3.1) we deduce (3.22) to finish the proof.

Theorem 3.6. *We have*

$$\frac{\chi^5(q)}{\chi(q^5)}=1+5q\frac{\psi^2(-q^5)}{\psi^2(-q)}. \tag{3.33}$$

The identity (3.33) was recorded by Ramanujan in the unorganized portion of his second notebook (Vol. II, p. 4 of [17]). Berndt (pp. 202–203. Entry 50(ii) of [6]) proved this by appealing to Ramanujan's modular equations. As shown by Baruah and Bhattacharyya (p. 2152, Theorem 2.2 of [4]), the identity easily follows from (3.2) and (2.3).

Theorem 3.7 (p. 276, equation (12.32) of [5]). *We have*

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} + q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) = 1. \tag{3.34}$$

Proof. Dividing both sides of (3.22) by $\varphi(q)\varphi(-q)$ and then using (2.6), we find that

$$\frac{\varphi(-q^5)}{\varphi(-q)} - \frac{\varphi(q^5)}{\varphi(q)} = \frac{4qf(-q^4)f(-q^{20})}{\varphi^2(-q^2)}. \tag{3.35}$$

With the help of (2.8), we rewrite the above identity as

$$\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)} = \frac{4qf(-q^4)\psi(q^{10})f(-q^{20})}{\psi(q^2)\varphi^2(-q^2)} \tag{3.36}$$

$$= \frac{4q\chi(-q^2)\psi(q^{10})f(-q^{20})}{\varphi^2(-q^2)}, \tag{3.37}$$

where in the last equality we used the fact

$$\frac{f(-q^4)}{\psi(q^2)} = \chi(-q^2)$$

readily deducible from the first and the last expressions of (2.3).

Again, replacing q by q^{10} in the second and the last expressions of (2.3), we have

$$\psi(q^{10}) = \frac{f^2(-q^{20})}{f(-q^{10})}. \tag{3.38}$$

From (3.36) and (3.38), we arrive at

$$\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)} = \frac{4q\chi(-q^2)f^3(-q^{20})}{\varphi^2(-q^2)f(-q^{10})}, \tag{3.39}$$

which can also be written with the help of (2.10) as

$$\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)} = \frac{4q\chi(-q^2)f(q^{10})f^2(-q^{20})}{\varphi^2(-q^2)\varphi(-q^{20})}. \tag{3.40}$$

But, replacing q by $-q^{20}$ in the second and the fourth expressions of (2.3), we have

$$\frac{f^2(-q^{20})}{\varphi(-q^{20})} = f(-q^{40}). \tag{3.41}$$

From (3.41) and (3.40), we find that

$$\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)} = \frac{4q\chi(-q^2)f(q^{10})f(-q^{40})}{\varphi^2(-q^2)}. \tag{3.42}$$

Employing (3.1), with q replaced by $-q^2$, in (3.42), we arrive at

$$\frac{\psi^2(-q^5)}{\psi^2(-q)}-\frac{\psi^2(q^5)}{\psi^2(q)}=\frac{\varphi^2(-q^{10})-\varphi^2(-q^2)}{q\varphi^2(-q^2)}, \tag{3.43}$$

which is easily seen to be equivalent to (3.34). □

Berndt (p. 285 of [5]) observed that the theta function identity in the following theorem is equivalent to Ramanujan's modular equation (4.1). He did not give a direct proof of this theta function identity. Here we provide a direct proof based on (3.1), (3.2), Lemmas 2.1–2.2 and Theorem 3.7.

Theorem 3.8 (p. 285 of [5]). *We have*

$$\varphi^2(q)\varphi^2(q^5)-\varphi^2(-q)\varphi^2(-q^5)-16q^3\psi^2(q^2)\psi^2(q^{10})=8qf^2(-q^2)f^2(q^{10}). \tag{3.44}$$

Proof. Replacing q by q^2 in (3.2) and then employing (2.3), we find that

$$\psi^2(q^2)-q^2\psi^2(q^{10})=\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\sqrt{\frac{\psi(q^2)}{\psi(q^{10})}}. \tag{3.45}$$

Again, with the aid of (2.3) we can recast (3.1) as

$$\varphi^2(q)-\varphi^2(q^5)=4q\sqrt{\frac{\varphi(q)}{\varphi(q^5)}}\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}. \tag{3.46}$$

Replacing q by $-q$ in (3.46), we obtain

$$\varphi^2(-q)-\varphi^2(-q^5)=-4q\sqrt{\frac{\varphi(-q)}{\varphi(-q^5)}}\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}. \tag{3.47}$$

Now, replacing q by q^5 in (2.9), we find that

$$\varphi^4(q^5)-\varphi^4(-q^5)=16q^5\psi^4(q^{10}). \tag{3.48}$$

From (2.9) and (3.48), we deduce that

$$\begin{aligned}&\varphi^2(q)\varphi^2(q^5)-\varphi^2(-q)\varphi^2(-q^5)-16q^3\psi^2(q^2)\psi^2(q^{10})\\&\quad=\frac{1}{2}\big\{16q(\psi^2(q^2)-q^2\psi^2(q^{10}))^2+(\varphi^2(-q)-\varphi^2(-q^5))^2\\&\qquad-(\varphi^2(q)-\varphi^2(q^5))^2\big\}.\end{aligned} \tag{3.49}$$

Using (3.45), (3.46) and (3.47) in (3.49), we find that

$$\begin{aligned}&\varphi^2(q)\varphi^2(q^5)-\varphi^2(-q)\varphi^2(-q^5)-16q^3\psi^2(q^2)\psi^2(q^{10})\\&\quad=8q\frac{f^5(-q^{10})}{f(-q^2)}\left\{\frac{\psi(q^2)}{\psi(q^{10})}+q\left(\frac{\varphi(-q)}{\varphi(-q^5)}-\frac{\varphi(q)}{\varphi(q^5)}\right)\right\}.\end{aligned} \tag{3.50}$$

Now, employing (2.6)–(2.8) in (3.34), we obtain

$$\frac{\varphi(q^5)\varphi(-q^5)}{\varphi(q)\varphi(-q)} + q\left(\frac{\varphi(q^5)\psi(q^{10})}{\varphi(q)\psi(q^2)} - \frac{\varphi(-q^5)\psi(q^{10})}{\varphi(-q)\psi(q^2)}\right) = 1. \tag{3.51}$$

We can rewrite the above identity as

$$\frac{\psi(q^2)}{\psi(q^{10})} + q\left(\frac{\varphi(-q)}{\varphi(-q^5)} - \frac{\varphi(q)}{\varphi(q^5)}\right) = \frac{\varphi(q)\varphi(-q)\psi(q^2)}{\varphi(q^5)\varphi(-q^5)\psi(q^{10})}. \tag{3.52}$$

Using (2.8) and (2.4) in (3.52), we obtain

$$\frac{\psi(q^2)}{\psi(q^{10})} + q\left(\frac{\varphi(-q)}{\varphi(-q^5)} - \frac{\varphi(q)}{\varphi(q^5)}\right) = \frac{f^3(-q^2)}{f^3(-q^{10})}. \tag{3.53}$$

With the help of (3.50) and (3.53), we finish the proof. □

Theorem 3.9 (Vol. I, p. 286 of [17], p. 364 of [7]). *We have*

$$\frac{\varphi^5(-q)}{\varphi(-q^5)} + 4\frac{f^5(-q)}{f(-q^5)} = 5\varphi^3(-q)\varphi(-q^5). \tag{3.54}$$

Proof. Multiplying both sides of (3.3) by $\varphi^3(q)/\varphi(q^5)$, we obtain

$$5\varphi^3(q)\varphi(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)} = 4\frac{\chi(q)\psi^2(-q)\varphi^3(q)\chi(q^5)}{\varphi(q^5)}. \tag{3.55}$$

Employing (2.3) four times in (3.55), we arrive at

$$5\varphi^3(q)\varphi(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)} = 4\frac{f^5(q)}{f(q^5)}. \tag{3.56}$$

Replacing q by $-q$ in (3.56) we readily deduce (3.54).

Theorem 3.10 (p. 364, Entry 16 of [7]). *We have*

$$\frac{\psi^5(q)}{\psi(q^5)} + \frac{\psi^5(-q)}{\psi(-q^5)} + 2\frac{f^5(-q^4)}{f(-q^{20})} = 4\frac{\psi^5(q^2)}{\psi(q^{10})}. \tag{3.57}$$

Proof. Multiplying (3.45) by $16q\psi^2(q^2)$, we obtain

$$16q\psi^4(q^2) - 16q^3\psi^2(q^2)\psi^2(q^{10}) = 16q\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\sqrt{\frac{\psi^5(q^2)}{\psi(q^{10})}}. \tag{3.58}$$

Again, multiplying (3.46) and (3.47) by $\varphi^2(q)$ and $\varphi^2(-q)$, respectively, and then subtracting the resulting identities, we deduce that

$$\varphi^4(q) - \varphi^4(-q) - \varphi^2(q)\varphi^2(q^5) + \varphi^2(-q)\varphi^2(-q^5)$$
$$= 4q\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\left\{\sqrt{\frac{\varphi^5(q)}{\varphi(q^5)}} + \sqrt{\frac{\varphi^5(-q)}{\varphi(-q^5)}}\right\}. \tag{3.59}$$

Subtracting (3.59) from (3.58), and using (2.9), we arrive at

$$\varphi^2(q)\varphi^2(q^5)-\varphi^2(-q)\varphi^2(-q^5)-16q^3\psi^2(q^2)\psi^2(q^{10})$$
$$=4q\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}\left\{4\sqrt{\frac{\psi^5(q^2)}{\psi(q^{10})}}-\sqrt{\frac{\varphi^5(q)}{\varphi(q^5)}}-\sqrt{\frac{\varphi^5(-q)}{\varphi(-q^5)}}\right\}. \tag{3.60}$$

From (3.44) and (3.60), we find that

$$4\sqrt{\frac{\psi^5(q^2)}{\psi(q^{10})}}-\sqrt{\frac{\varphi^5(q)}{\varphi(q^5)}}-\sqrt{\frac{\varphi^5(-q)}{\varphi(-q^5)}}=2\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{3.61}$$

Multiplying (3.61) by $\sqrt{\varphi^5(q)/\varphi(q^5)}$ and using also (2.3), (2.6) and (2.8) we arrive at the required identity. □

Theorem 3.11 (p. 364, Entry 15 of [7]). *We have*

$$\frac{\psi^5(-q^5)}{\psi(-q)}-\frac{\psi^5(q^5)}{\psi(q)}=2q\frac{f^5(-q^{20})}{f(-q^4)}+4q^3\frac{\psi^5(q^{10})}{\psi(q^2)}. \tag{3.62}$$

Proof. Using (2.3) in (3.3), we find that

$$\varphi^2(q)-5\varphi^2(q^5)=-4\sqrt{\frac{\varphi(q^5)}{\varphi(q)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{3.63}$$

Replacing q by $-q$, we obtain

$$\varphi^2(-q)-5\varphi^2(-q^5)=-4\sqrt{\frac{\varphi(-q^5)}{\varphi(-q)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{3.64}$$

Multiplying (3.63) and (3.64) by $\varphi^2(q^5)$ and $\varphi^2(-q^5)$, respectively, and then subtracting the resulting second identity from the first, we deduce that

$$\varphi^2(q)\varphi^2(q^5)-\varphi^2(-q)\varphi^2(-q^5)-80q^5\psi^4(q^{10})$$
$$=4\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}\left\{\sqrt{\frac{\varphi^5(-q^5)}{\varphi(-q)}}-\sqrt{\frac{\varphi^5(q^5)}{\varphi(q)}}\right\}. \tag{3.65}$$

Now, replacing q by q^2 in (3.4), we have

$$\psi^2(q^2)-5q^2\psi^2(q^{10})=\frac{\varphi^2(-q^2)}{\chi(-q^2)\chi(-q^{10})}. \tag{3.66}$$

With the help of (2.3), we rewrite (3.66) as

$$\psi^2(q^2)-5q^2\psi^2(q^{10})=\sqrt{\frac{\psi(q^{10})}{\psi(q^2)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \tag{3.67}$$

Multiplying (3.67) by $16q^3\psi^2(q^{10})$, we find that

$$16q^3\psi^2(q^2)\psi^2(q^{10}) - 80q^5\psi^4(q^{10}) = 16q^3\sqrt{\frac{\psi^5(q^{10})}{\psi(q^2)}}\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}. \quad (3.68)$$

From (3.65) and (3.68), we deduce that

$$\begin{aligned}&-\varphi^2(q)\varphi^2(q^5) + \varphi^2(-q)\varphi^2(-q^5) + 16q^3\psi^2(q^2)\psi^2(q^{10})\\ &= 4\sqrt{\frac{f^5(-q^2)}{f(-q^{10})}}\left\{4q^3\sqrt{\frac{\psi^5(q^{10})}{\psi(q^2)}} + \sqrt{\frac{\varphi^5(q^5)}{\varphi(q)}} - \sqrt{\frac{\varphi^5(-q^5)}{\varphi(-q)}}\right\}.\end{aligned} \quad (3.69)$$

From (3.44) and (3.69), we arrive at

$$4q^3\sqrt{\frac{\psi^5(q^{10})}{\psi(q^2)}} + \sqrt{\frac{\varphi^5(q^5)}{\varphi(q)}} - \sqrt{\frac{\varphi^5(-q^5)}{\varphi(-q)}} = -2q\sqrt{\frac{f^5(-q^{10})}{f(-q^2)}}. \quad (3.70)$$

Multiplying both side of (3.70) by $\sqrt{\varphi^5(q^5)/\varphi(q)}$ and using (2.3), we find that

$$4q^3\frac{\psi^5(q^5)}{\psi(q)} + \frac{\varphi^5(q^5)}{\varphi(q)} - \frac{\varphi^5(-q^{10})}{\varphi(-q^2)} = -2q\frac{f^5(q)}{f(q^5)}. \quad (3.71)$$

Multiplying both sides of (3.71) by $(\psi^5(q^5)\varphi(q))/(\varphi^5(q^5)\psi(q))$ and rearranging the results by employing some identities in Lemmas 2.1 and 2.2, we deduce (3.62) to finish the proof. □

Theorem 3.12 (p. 363, Entry 14 of [7]). *We have*

$$5\frac{\varphi^2(q)}{\varphi^2(q^5)} = \frac{\frac{\varphi^5(q)}{\varphi(q^5)} + 4\frac{\psi^5(q)}{\psi(q^5)}}{\varphi(q)\varphi^3(q^5) + 4q^2\psi(q)\psi^3(q^5)}. \quad (3.72)$$

Proof. From (3.3) and (3.4), we obtain

$$\frac{\varphi^2(q) - 5\varphi^2(q^5)}{\psi^2(q^2) - 5q^2\psi^2(q^{10})} = -4\frac{\psi^2(-q)\chi(q)\chi(-q^2)\chi(q^5)\chi(-q^{10})}{\varphi^2(-q^2)}. \quad (3.73)$$

Employing Lemmas 2.1 and 2.2 on the right-hand side of (3.73), we find that

$$\frac{\varphi^2(q) - 5\varphi^2(q^5)}{\psi^2(q^2) - 5q^2\psi^2(q^{10})} = -4\frac{\psi(q)\varphi(q^5)}{\varphi(q)\psi(q^5)}. \quad (3.74)$$

Thus,

$$\varphi^2(q) - 5\varphi^2(q^5) = -4\frac{\psi^2(q^2)\psi(q)\varphi(q^5)}{\varphi(q)\psi(q^5)} + 20q^2\frac{\psi^2(q^{10})\varphi(q^5)\psi(q)}{\varphi(q)\psi(q^5)}. \quad (3.75)$$

Employing (2.8) in (3.75), we arrive at

$$\varphi^2(q) - 5\varphi^2(q^5) = -4\frac{\psi^5(q)\varphi(q^5)}{\varphi^3(q)\psi(q^5)} + 20q^2\frac{\psi^3(q^5)\psi(q)}{\varphi(q)\varphi(q^5)}. \quad (3.76)$$

Multiplying both sides of (3.76) by $\varphi^3(q)/\varphi(q^5)$, we deduce that

$$\frac{\varphi^5(q)}{\varphi(q^5)}+4\frac{\psi^5(q)}{\psi(q^5)}=5\varphi^3(q)\varphi(q^5)+20q^2\psi(q)\psi^3(q^5)\frac{\varphi^2(q)}{\varphi^2(q^5)}, \tag{3.77}$$

which is obviously equivalent to (3.72). □

4. Modular equations

In this section, we find alternative proofs of Ramanujan's modular equations of degree 5 by using the theta function identities established in the previous section. As we mentioned earlier, Berndt proved some of these equations by a method of parametrizations. Some of the proofs given in this are possibly closer to the proofs of Ramanujan. Throughout this section, we suppose that β has degree 5 over α and $m = z_1/z_5$ is the corresponding multiplier.

Theorem 4.1 (p. 280, Entry 13(i) of [5]). *We have*

$$(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}=1. \tag{4.1}$$

Proof. Transcribing (3.44), by using (2.13), (2.14), (2.20) and (2.21), we easily deduce (4.1).

Baruah [1] has also found a different proof based on the identity (3.22).

Theorem 4.2 (p. 280, Entry 13(ii) of [5]). *We have*

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8}-\left(\frac{(1-\alpha)^5}{1-\beta}\right)=1+2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{4.2}$$

Proof. Transcribing (3.57), by using (2.18) – (2.20) and (2.23), we readily deduce (4.2).

Theorem 4.3 (p. 280, Entry 13(iii) of [5]). *We have*

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8}-\left(\frac{\beta)^5}{\alpha}\right)^{1/8}=1+2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}. \tag{4.3}$$

Proof. Transcribing (3.62) by means of (2.18)–(2.20) and (2.23), we deduce (4.3).

In fact, (4.2) and (4.3) are reciprocal to each other in the sense of Entry 24(v) (p. 216 of [5]).

Theorem 4.4 (p. 280, Entry 13(iv) of [5]). *We have*

$$m=1+2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}. \tag{4.4}$$

and

$$\frac{5}{m}=1+2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{4.5}$$

Proof. Transcribing (3.1) with the help of (2.21) and (2.25), we arrive at (4.4).

Replacing q by $-q$ in (3.54) and then transcribing the resulting identity with the aid of (2.13) and (2.22), we easily deduce (4.5).

Note that (4.5) is also the reciprocal of (4.4) and vice-versa.

Theorem 4.5 (p. 280, Entry 13(v) of [5]). *We have*

$$m = \frac{1 + ((1-\beta)^5/(1-\alpha))^{1/8}}{1 + \{(1-\alpha)^3(1-\beta)\}^{1/8}} = \frac{1 - (\beta^5/\alpha)^{1/8}}{1 - (\alpha^3\beta)^{1/8}}. \tag{4.6}$$

As shown by Bernd pp. 282–283 of [5], the above modular equations can be derived by transcribing, with the help of Lemma 2.5, the theta function identities

$$\frac{\varphi^2(-q) - \varphi^2(-q^5)}{\varphi^2(q) - \varphi^2(q^5)} = -\frac{\chi(-q)f(q^5)}{\chi(q)f(-q^5)} \tag{4.7}$$

and

$$\frac{\psi^2(q^2) - q^2\psi^2(q^{10})}{\varphi^2(q) - \varphi^2(q^5)} = \frac{\varphi(-q^{10})f(-q^{10})}{4q\chi(q)\chi(-q^2)f(-q^5)f(-q^{20})}, \tag{4.8}$$

both readily deducible from (3.1) and (3.2).

Theorem 4.6 (p. 280, Entry 13(vi) of [5]). *We have*

$$\frac{5}{m} = \frac{1 + \left(\alpha^5/\beta\right)^{1/8}}{1 + (\alpha\beta^3)^{1/8}} = \frac{1 - \left((1-\alpha)^5/(1-\beta)\right)^{1/8}}{1 - \{(1-\alpha)(1-\beta)^3\}^{1/8}}. \tag{4.9}$$

These modular equations are the reciprocals of the respective modular equations in the previous theorem. Here we also offer an alternative proof.

Proof. Transcribing (3.72) by employing (2.13) and (2.18), we easily deduce the first equality of (4.9).

To prove the other assertion, first we replace q by $-q$, in (3.3) to obtain

$$\varphi^2(-q) - 5\varphi^2(-q^5) = -4\frac{\chi(-q^5)\varphi^2(-q)}{\chi^5(-q)}. \tag{4.10}$$

Next, from (3.3) and (4.10), we have

$$\frac{\varphi^2(-q) - 5\varphi^2(-q^5)}{\varphi^2(q) - 5\varphi^2(q^5)} = \frac{\chi(-q)\chi(-q^5)\psi^2(q)}{\chi(q)\chi(q^5)\psi^2(-q)}. \tag{4.11}$$

Transcribing the above identity by employing (2.13), (2.14), (2.18), (2.19), (2.25) and (2.26), we find that

$$\frac{(1-\alpha)^{1/2} - 5/m(1-\beta)^{1/2}}{1 - 5/m} = \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}. \tag{4.12}$$

Dividing both sides of the above identity by $((1-\beta)/(1-\alpha))^{1/8}$, we easily deduce

$$\frac{5}{m} = \frac{1 - \left((1-\alpha)^5/(1-\beta)\right)^{1/8}}{1 - \{(1-\alpha)(1-\beta)^3\}^{1/8}} \tag{4.13}$$

to finish the proof. □

Theorem 4.7 (p. 280, Entry 13(vii) of [5]). *We have*

$$(\alpha\beta^3)^{1/8} + \{(1-\alpha)(1-\beta)^3\}^{1/8} = 1 - 2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24}$$
$$= (\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8}$$
$$= \left(\frac{1+(\alpha\beta)^{1/2}+(1-\alpha)(1-\beta)^{1/2}}{2}\right)^{1/2}. \tag{4.14}$$

Proof. Replacing q by $q^{1/4}$ in (3.22), we find that

$$\varphi(q^{5/4})\varphi(-q^{1/4}) - \varphi(-q^{5/4})\varphi(q^{1/4}) = -4q^{1/4}f(-q)f(-q^5). \tag{4.15}$$

Transcribing this equation by using (2.13), (2.14) and (2.24), we obtain

$$\alpha^{1/4} - \beta^{1/4} = 2^{2/3}(\alpha\beta)^{1/24}((1-\alpha)(1-\beta))^{1/6}. \tag{4.16}$$

The reciprocal of the above equation is given by

$$(1-\beta)^{1/4} - (1-\alpha)^{1/4} = 2^{2/3}(\alpha\beta)^{1/6}((1-\alpha)(1-\beta))^{1/24}. \tag{4.17}$$

Berndt *et al* (Lemma 9.1, p. 20 of [9]) have also given a proof of (4.17) by using the method of parametrizations. They used this to prove some results on Ramanujan's famous forty identities for the Rogers–Ramanujan functions.

Multiplying both sides of (4.17) by $((1-\alpha)(1-\beta))^{1/8}$, we find that

$$((1-\beta)^3(1-\alpha))^{1/8} - ((1-\alpha)^3(1-\beta))^{1/8} = 2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6}. \tag{4.18}$$

The reciprocal of this equation is given by

$$(\alpha^3\beta)^{1/8} - (\beta^3\alpha)^{1/8} = 2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6}. \tag{4.19}$$

From (4.18) and (4.19), we deduce that

$$((1-\beta)^3(1-\alpha))^{1/8} - ((1-\alpha)^3(1-\beta))^{1/8} = (\alpha^3\beta)^{1/8} - (\beta^3\alpha)^{1/8}. \tag{4.20}$$

We rewrite the above identity as

$$(\alpha\beta^3)^{1/8} + \{(1-\alpha)(1-\beta)^3\}^{1/8} = (\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8}. \tag{4.21}$$

Now, dividing both sides of (4.16) by $(\alpha\beta)^{1/24}$, we obtain

$$\left(\frac{\alpha^5}{\beta}\right)^{1/24} - \left(\frac{\beta^5}{\alpha}\right)^{1/24} = 2^{2/3}((1-\alpha)(1-\beta))^{1/6}. \tag{4.22}$$

Multiplying both sides of the above identity by $2^{1/3}\left((1-\alpha)^5/(1-\beta)\right)^{1/24}$, we find that

$$2^{1/3}\left(\frac{(\alpha(1-\alpha))^5}{\beta(1-\beta)}\right)^{1/24} - 2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24} = 2((1-\alpha)^3(1-\beta))^{1/8}. \tag{4.23}$$

Again, multiplying both sides of (4.21) by $2^{1/3}((1-\beta)^5/(1-\alpha))^{1/24}$, we arrive at

$$2^{1/3}\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{(\beta(1-\beta))^5}{\alpha(1-\alpha)}\right)^{1/24} = 2((-\beta)^3(1-\alpha))^{1/8}. \tag{4.24}$$

The reciprocal of (4.24) is given by

$$2^{1/3}\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{(\alpha(1-\alpha))^5}{\beta(1-\beta)}\right)^{1/24} = 2(\alpha^3\beta)^{1/8}. \tag{4.25}$$

Adding (4.23) and (4.25), we deduce that

$$\begin{aligned}&2^{1/3}\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}\\ &\quad = 2\{(\alpha^3\beta)^{1/8} + (1-\alpha)^3(1-\beta))^{1/8}\}.\end{aligned} \tag{4.26}$$

Dividing both sides by $2^{-1/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/24}$, we obtain

$$\left(\frac{(\alpha(1-\beta))^5}{\beta(1-\alpha)}\right)^{1/24} = 2^{2/3} - \left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}. \tag{4.27}$$

Employing (4.27) in (4.26), we find that

$$(\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8} = 1 - 2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24}. \tag{4.28}$$

Now, multiplying both sides of (4.27) by $4^{1/3}((\beta(1-\alpha))^5/\alpha(1-\beta))^{1/24}$, we have

$$4^{1/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6} + 4^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/12} = 4^{2/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}. \tag{4.29}$$

The above identity can also be written as

$$\begin{aligned}1-(16\alpha\beta(1-\alpha)(1-\beta))^{1/6} &= 1 + 4^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/12}\\ &\quad - 2.2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}\\ &= \left(1 - 2^{1/3}\left(\frac{(\beta(1-\alpha))^5}{\alpha(1-\beta)}\right)^{1/24}\right)^2.\end{aligned} \tag{4.30}$$

Now, we recast (4.1) as

$$1-(16\alpha\beta(1-\alpha)(1-\beta))^{1/6} = \left(\frac{1+(\alpha\beta)^{1/2} + ((1-\alpha)(1-\beta))^{1/2}}{2}\right)^{1/2}. \tag{4.31}$$

From (4.30) and (4.31), we arrive at

$$1-2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24}=\left(\frac{1+(\alpha\beta)^{1/2}+(1-\alpha)(1-\beta)^{1/2}}{2}\right)^{1/2}. \tag{4.32}$$

Thus, from (4.21), (4.28) and (4.32), we obtain (4.14) to finish the proof.

Theorem 4.8 (p. 366, Entry 20 of [7]). *We have*

$$(\alpha\beta^3)^{1/8}+\{(1-\alpha)(1-\beta)^3\}^{1/8}=\sqrt{1-(\alpha\beta(1-\alpha)(1-\beta))^{1/6}}. \tag{4.33}$$

Proof. This identity follows readily from (4.14) and (4.31).

Theorem 4.9 (p. 280, Entry 13(ix) of [5]). *We have*

$$1+4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}=\frac{m}{2}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}) \tag{4.34}$$

and

$$1+4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}=\frac{5}{2m}(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}). \tag{4.35}$$

Proof. From (4.51), we notice that

$$m^2+5=2m(2+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}). \tag{4.36}$$

Again, from (4.4), we obtain

$$m-1=2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}. \tag{4.37}$$

Squaring both sides of the above equation, we find that

$$m^2+1=2m+2^{8/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}. \tag{4.38}$$

Using (4.38) in (4.36), we obtain (4.34).

Taking reciprocal, we immediately arrive at (4.35).

Theorem 4.10 (p. 280, Entry 13(x) of [5]). *We have*

$$\begin{aligned}\{\alpha(1-\beta)\}^{1/4}+\{\beta(1-\alpha)\}^{1/4}&=4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}\\&=m\{\alpha(1-\alpha)\}^{1/4}+\{\beta(1-\beta)\}^{1/4}\\&=\{\alpha(1-\alpha)\}^{1/4}+\frac{5}{m}\{\beta(1-\beta)\}^{1/4}.\end{aligned} \tag{4.39}$$

Proof. From the two identities in Theorem 1.12, we find that

$$m(\alpha(1-\alpha))^{1/4}+(\beta(1-\beta))^{1/4}=(\beta(1-\alpha))^{1/4}+(\alpha(1-\beta))^{1/4} \tag{4.40}$$

and

$$\frac{5}{m}(\beta(1-\beta))^{1/4}+(\alpha(1-\alpha))^{1/4}=(\alpha(1-\beta))^{1/4}+(\beta(1-\alpha))^{1/4}. \tag{4.41}$$

Now, replacing q by $-q$ in (3.2) and then transcribing the resulting identity with the help of (2.13), (2.19), (2.22) and (2.25), we deduce that

$$4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}=m\{\alpha(1-\alpha))\}^{1/4}+\{\beta(1-\beta)\}^{1/4}. \tag{4.42}$$

By means of (4.40), (4.41) and (4.42), we readily finish the proof of the theorem.

Theorem 4.11 (p. 280, Entry 13(xi) of [5]). *We have*

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8}+\left(\frac{\beta^5}{\alpha}\right)^{1/8}=m\left(\frac{1+(\alpha\beta)^{1/2}+(1-\alpha)(1-\beta)^{1/2}}{2}\right)^{1/2} \tag{4.43}$$

and

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8}+\left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8}=\frac{5}{m}\left(\frac{1+(\alpha\beta)^{1/2}+(1-\alpha)(1-\beta)^{1/2}}{2}\right)^{1/2}. \tag{4.44}$$

Proof. From (4.6), we find that

$$m\{1+((1-\alpha)^3(1-\beta))^{1/8}\}=1+\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8} \tag{4.45}$$

and

$$m\{1-(\alpha^3\beta)^{1/8}\}=1-\left(\frac{\beta^5}{\alpha}\right)^{1/8}. \tag{4.46}$$

Subtracting (4.46) from (4.45), we obtain

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8}+\left(\frac{\beta^5}{\alpha}\right)^{1/8}=m\{(\alpha^3\beta)^{1/8}+\{(1-\alpha)^3(1-\beta)\}^{1/8}\}. \tag{4.47}$$

Using (4.14) in (4.47), we readily deduce (4.43).

The identity (4.44) is the reciprocal of (4.43).

Theorem 4.12 (p. 280, Entry 13(xii) of [5]). *We have*

$$m=\left(\frac{\beta}{\alpha}\right)^{1/4}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{4.48}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \tag{4.49}$$

Proof. Transcribing (3.34) by employing (2.15), (2.18) and (2.19), we easily deduce (4.48). Taking reciprocal of (4.48) we obtain (4.49).

Ramanujan also recorded these two modular equations in his Lost Notebook (p. 351 of [18]).

Theorem 4.13 (p. 280, Entry 13(xiii) of [5]). *We have*

$$m - \frac{5}{m} = \frac{4((\alpha\beta)^{1/2} - \{(1-\alpha)(1-\beta)\}^{1/2})}{(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}/2)^{1/2}} \tag{4.50}$$

and

$$m + \frac{5}{m} = 2(2 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}). \tag{4.51}$$

Proof. Cubing both sides of the identity (4.22), we find that

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \left(\frac{\beta^5}{\alpha}\right)^{1/8} - 3.2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6} = 4((1-\alpha)(1-\beta))^{1/2}. \tag{4.52}$$

Taking reciprocal of this equation, we obtain

$$\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{(1-\beta)}\right)^{1/8} - 3.2^{2/3}(\alpha\beta(1-\alpha)(1-\beta))^{1/6} = 4(\alpha\beta)^{1/2}. \tag{4.53}$$

Subtracting (4.53) from (4.52), we deduce that

$$\begin{aligned}&\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{(1-\beta)}\right)^{1/8} + \left(\frac{\beta^5}{\alpha}\right)^{1/8} - \left(\frac{\alpha^5}{\beta}\right)^{1/8}\\ &= 4\{(\alpha\beta)^{1/2} - ((1-\alpha)(1-\beta))\}^{1/2}.\end{aligned} \tag{4.54}$$

Now, subtracting (4.44) from (4.43), we find that

$$\begin{aligned}&\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{1/8} + \left(\frac{\beta^5}{\alpha}\right)^{1/8} - \left(\left(\frac{(1-\alpha)^5}{(1-\beta)}\right)^{1/8} + \left(\frac{\alpha^5}{\beta}\right)^{1/8}\right)\\ &= \left(m - \frac{5}{m}\right)\left(\frac{1 + (\alpha\beta)^{1/2} + (1-\alpha)(1-\beta)^{1/2}}{2}\right)^{1/2}.\end{aligned} \tag{4.55}$$

From (4.54) and (4.55), we easily deduce (4.50).

Next, transcribing (3.20) by employing (2.13) and (2.21), we find that

$$\frac{6m - m^2 - 5}{4m} = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}. \tag{4.56}$$

Thus,

$$m + \frac{5}{m} = 2\{2(1 - \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}\} + 2. \tag{4.57}$$

Employing (4.1) in (4.57), we deduce (4.51) to finish the proof.

Remark 4.14. We refer the readers to [14], [8], [2] and [3] for beautiful partition-theoretic interpretations of some of the theta function identities and modular equations of degree 5 discussed in this paper.

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