Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No. 4, November 2012, pp. 561–571.

On dominator colorings in graphs

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MS received 30 March 2011; revised 30 September 2011

Abstract. A dominator coloring of a graph G is a proper coloring of G in which every vertex dominates every vertex of at least one color class. The minimum number of colors required for a dominator coloring of G is called the dominator chromatic number of G and is denoted by $\chi_d(G)$. In this paper we present several results on graphs with $\chi_d(G) = \chi(G)$ and $\chi_d(G) = \gamma(G)$ where $\chi(G)$ and $\gamma(G)$ denote respectively the chromatic number and the domination number of a graph G. We also prove that if $\mu(G)$ is the Mycielskian of G, then $\chi_d(G) + 1 \le \chi_d(\mu(G)) \le \chi_d(G) + 2$.

Keywords. Dominator coloring; dominator chromatic number; chromatic number; domination number.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by $n = |V|$ and $m = |E|$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

Graph coloring and domination are two major areas in graph theory that have been well studied. An excellent treatment of fundamentals of domination is given in the book by Haynes *et al*. [13] and survey papers on several advanced topics are given in the book edited by Haynes *et al*. [14].

Let $G = (V, E)$ be a graph. Let $v \in V$. The *degree* of a vertex v in a graph G is defined to be the number of edges incident with v and is denoted by deg v. A vertex of degree zero in G is an *isolated vertex* and a vertex of degree one is a *pendant vertex* or a *leaf*. Any vertex which is adjacent to a pendant vertex is called a *support vertex*. The *open neighborhood* $N(v)$ and the *closed neighborhood* $N[v]$ of v are defined by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of V is said to be an *independent set* if no two vertices in S are adjacent. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in G. A subset S of V is called a *dominating set* of G if every vertex in $V - S$ is adjacent to a vertex in S, in which case we also say that S is a *dominator* of $V - S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G.

A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors required for a proper coloring of G. The *clique number* $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. For a set S of vertices of G, the *induced subgraph* is the maximal subgraph of G with vertex set S and is denoted by $\langle S\rangle$. Thus two vertices of S are adjacent in $\langle S\rangle$ if and only if they are adjacent in G. A graph G is called *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

A graph G is called a *split graph* if its vertex set can be partitioned into a clique and an independent set. The *corona* $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined to be the graph G obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2.

Hedetniemi *et al*. [15, 16] introduced the concepts of dominator partition and dominator coloring of a graph.

DEFINITION 1.1

A vertex $v \in V$ is a dominator of a set $S \subseteq V$ if v dominates every vertex in S. A partition $\pi = \{V_1, V_2, \ldots, V_k\}$ is called a dominator partition if every vertex $v \in V$ is a dominator of at least one V_i. The dominator partition number $\pi_d(G)$ equals the minimum k such that G has a dominator partition of order k. If we further require that π be a proper coloring of G, then we have a dominator coloring of G. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G.

Since every vertex is a dominator of itself, the partition $\{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ into singleton sets is a dominator coloring. Thus, every graph of order n has a dominator coloring of order n and therefore the dominator chromatic number $\chi_d(G)$ is well defined. Gera *et al*. [12] also studied this concept. Some basic results on dominator colorings are given in [5, 10–12]. In this paper we present further results on dominator colorings.

We need the following theorems.

Theorem 1.2 [10]. *Let G be a connected graph of order $n \geq 2$. Then $\chi_d(G) = 2$ if and only if G is a complete bipartite graph of the form $K_{a,b}$, where $1 \leq a \leq b \leq n$ and $a + b = n$.*

Theorem 1.3 [10]. *Let G be a connected graph of order n. Then $\chi_d(G) = n$ if and only if G is the complete graph K_n.*

Theorem 1.4 [5]. *Let T be a tree of order $n \geq 2$. Then $\gamma(T) + 1 \leq \chi_d(T) \leq \gamma(T) + 2$.*

Theorem 1.5 [10]. *For the cycle C_n, we have*

$$\chi_d(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 4 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n = 5 \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{otherwise.} \end{cases}$$

Theorem 1.6 [12]. *For the path P_n, $n \geq 2$, we have*

$$\chi_d(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n = 2, 3, 4, 5, 7 \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{otherwise.} \end{cases}$$

If $\{V_1, V_2, \ldots, V_{\chi_d}\}$ is a χ_d-coloring of G and if $v_i \in V_i$, then $S = \{v_1, v_2, \ldots, v_{\chi_d}\}$ is a dominating set of G. Also if D is a γ-set of G, then $\mathcal{C} \cup \{\{v\} : v \in D\}$ where $\mathcal{C}$ is a proper coloring of $G - D$ gives a dominator coloring of G. These observations lead to the following bounds for $\chi_d(G)$.

Theorem 1.7 [10]. *Let G be a connected graph. Then* $\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$.

COROLLARY 1.8

For any bipartite graph G, $\gamma(G) \leq \chi_d(G) \leq \gamma(G) + 2$.

PROPOSITION 1.9 [12]

For a connected graph G of order $n \geq 3$, $\chi_d(G) \leq n - \beta_0(G) + 1$, and this bound is sharp.

PROPOSITION 1.10 [12]

If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ with $k \geq 2$, then $\max\limits_{i \in \{1,2,\ldots,k\}} \chi_d(G_i) + k - 1 \leq \chi_d(G) \leq \sum_{i=1}^{k} \chi_d(G_i)$, and these bounds are sharp.

Observation 1.11. For any graph G, $\omega(G) \leq \chi(G) \leq \chi_d(G)$.

2. Basic results

Theorem 1.3 shows that the complete graph K_n is the only connected graph of order n with $\chi_d(G) = n$. We start with a simple generalization of this result.

PROPOSITION 2.1

Let G be a graph of order n. Then $\chi_d(G) = n$ if and only if $G = K_a \cup (n - a)K_1$, where $1 \leq a \leq n$.

Proof. Suppose $\chi_d(G) = n$. Clearly every component of G is complete. If G contains two nontrivial components G_1 and G_2, choose $u \in V(G_1)$ and $v \in V(G_2)$. Then $\{\{u, v\}\} \cup \{\{x\} : x \in V(G) - \{u, v\}\}$ is a dominator coloring of G, which is a contradiction. Hence $G = K_a \cup (n - a)K_1$, where $1 \leq a \leq n$.

The converse is obvious. □

We now proceed to characterize graphs with $\chi_d(G) = n - 1$.

Theorem 2.2. *Let G be a connected graph of order n. Then $\chi_d(G) = n - 1$ if and only if one of the following holds*:

(i) *$G \neq K_n$ and K_{n-1} is a subgraph of G.*

(ii) $V(G) = V_1 \cup \{u, v\}$, *where* $\langle V_1 \rangle = K_{n-2}$, deg $u = 1$, $uv \in E(G)$ *and* v *is nonadjacent to at least one vertex in* V_1.

Proof. Let G be a connected graph of order n with $\chi_d(G) = n - 1$ and let $V(G) = \{v_1, v_2, \ldots, v_n\}$. By Proposition 1.9, we have $\chi_d(G) \leq n - \beta_0(G) + 1$ and hence $\beta_0(G) = 2$. If there exist three disjoint β_0-sets, say $\{v_1, v_2\}, \{v_3, v_4\}$ and $\{v_5, v_6\}$, then $\{\{v_1, v_2\}, \{v_3, v_4\}\} \cup \{\{v_i\} : 5 \leq i \leq n\}$ is a dominator coloring of G and hence $\chi_d(G) \leq n - 2$, which is a contradiction. Hence there exist at most two disjoint β_0-sets in G. We consider two cases.

Case i. Any two β_0-sets in G are not disjoint.

Let $\{v_1, v_2\}$ be a β_0-set in G. Clearly the induced subgraph $H = \langle\{v_3, v_4, \ldots, v_n\}\rangle$ is complete. We now claim that either v_1 or v_2 is adjacent to every vertex of H.

Suppose there exist $v_i, v_j \in V(H)$ such that $v_1 v_i, v_2 v_j \notin E(G)$. Since $\beta_0(G) = 2$, it follows that $i \neq j$ and hence $\{v_1, v_i\}, \{v_2, v_j\}$ are two disjoint β_0-sets in G, which is a contradiction. Hence we may assume that v_2 is adjacent to every vertex in H. Hence $\langle V(H) \cup \{v_2\}\rangle$ is isomorphic to K_{n-1} and G is of the form (i).

Case ii. There exist two disjoint β_0-sets in G, say $\{v_1, v_2\}$ and $\{v_3, v_4\}$.

Since $\chi_d(G) = n - 1$, it follows that $\mathcal{C} = \{\{v_1, v_2\}, \{v_3, v_4\}\} \cup \{\{v_i\} : 5 \leq i \leq n\}$ is not a dominator coloring of G. We may assume without loss of generality that the vertex v_1 does not dominate any color class. Hence deg $v_1 = 1$ and let $v_1 v_3 \in E(G)$. Now $H_1 = \langle\{v_2, v_4, v_5, \ldots, v_n\}\rangle = K_{n-2}$ and hence G is of the form (ii).

Conversely, if a graph G satisfies condition (i) or (ii), it is easy to see that $\chi_d(G) = n - 1$. □

Theorem 2.3. *Let G be a graph of order n. Then* $\chi_d(G) = n - 1$ *if and only if one of the following is true.*

(i) *G contains exactly one nontrivial component* G_1 *of order* n_1 *with* $\chi_d(G_1) = n_1 - 1$.
(ii) *G contains exactly two nontrivial components say* G_1 *and* G_2, *where* $G_1 = K_{n_1}, n_1 \geq 2$ *and* $G_2 = K_2$.

Proof. Suppose $\chi_d(G) = n - 1$ and let $G_1, G_2, \ldots, G_k$ be the set of all components of G of order $n_1, n_2, \ldots, n_k$ respectively. Now we claim that G contains at most one noncomplete component. Suppose G_1 and G_2 are noncomplete. Then $\chi_d(G_1) \leq n_1 - 1$ and $\chi_d(G_2) \leq n_2 - 1$ and hence $\chi_d(G) \leq \sum_{i=1}^{k} \chi_d(G_i) < n - 1$, which is a contradiction.

Case i. G_1 is not complete.

Then $G_2, G_3, \ldots, G_k$ are complete. Let u and v be two nonadjacent vertices in G_1. If $n_2 \geq 2$, let $w \in V(G_2)$. Then $\{\{u, v, w\}\} \cup \{\{v_i\} : v_i \notin \{u, v, w\}\}$ is a dominator coloring of G and hence $\chi_d(G) < n - 1$, which is a contradiction. Thus $G_2 = G_3 = \cdots = G_k = K_1$. Further since $\chi_d(G) = n - 1$, it follows that $\chi_d(G_1) = n_1 - 1$.

Case ii. Every component of G is complete.

It follows from Proposition 2.1 that G contains at least two nontrivial components, say G_1 and G_2. We now claim that $G_3 = G_4 = \cdots = G_k = K_1$. If $n_3 \geq 2$, choose $u \in V(G_1), v \in V(G_2)$ and $w \in V(G_3)$. Then $\{\{u, v, w\}\} \cup \{\{v_i\} : v_i \notin \{u, v, w\}\}$ is a dominator coloring of G and hence $\chi_d(G) < n - 1$, which is a contradiction. Hence $G_3 = G_4 = \cdots = G_k = K_1$. Now if $n_1 \geq 3$ and $n_2 \geq 3$, choose $u, v \in V(G_1)$ and

$u', v' \in V(G_2)$. Then $\{\{u, u'\}, \{v, v'\}\} \cup \{\{v_i\} : v_i \notin \{u, u', v, v'\}\}$ is a dominator coloring of G and hence $\chi_d(G) < n - 1$, which is a contradiction. Thus G is of the form (ii).

The converse is obvious. □

The next result concerns the complement $\bar{G}$ of a graph G, where $V(\bar{G}) = V(G)$ and two vertices u and v are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Theorem 2.4. *Let G be a connected graph of order $n \geq 2$. Then $4 \leq \chi_d(G) + \chi_d(\bar{G}) \leq 2n$. Further $\chi_d(G) + \chi_d(\bar{G}) = 4$ if and only if $G = K_2$ and $\chi_d(G) + \chi_d(\bar{G}) = 2n$ if and only if $G = K_n$.*

Proof. Since $2 \leq \chi_d(G) \leq n$, the inequalities are trivial. Also $\chi_d(G) + \chi_d(\bar{G}) = 2n$ if and only if $\chi_d(G) = \chi_d(\bar{G}) = n$ and hence it follows from Theorem 1.3 that $G = K_n$. Also $\chi_d(G) + \chi_d(\bar{G}) = 4$ if and only if $\chi_d(G) = \chi_d(\bar{G}) = 2$ and hence it follows from Theorem 1.2 that $G = K_2$.

The converse is obvious. □

PROPOSITION 2.5

Let G be a connected graph of order n. Then $\chi_d(G) + \chi_d(\bar{G}) = 2n - 1$ if and only if $G = K_n - e$.

Proof. Suppose $\chi_d(G) + \chi_d(\bar{G}) = 2n - 1$. If $\chi_d(G) = n$, then it follows from Theorem 1.3 that $G = K_n$ and $\chi_d(\bar{G}) = n$, which is a contradiction. Hence $\chi_d(G) = n - 1$ and $\chi_d(\bar{G}) = n$. Hence it follows from Proposition 2.1 that $\bar{G} = K_2 \cup (n - 2)K_1$, so that $G = K_n - e$.

The converse is obvious. □

Observation 2.6. In [11] it has been conjectured that for the n-dimensional hypercube Q_n, $\chi_d(Q_n) = 2 + 2^{n-2}$. This conjecture is false. It has been proved in [1] that $\gamma(Q_7) = 16$. Hence $\chi_d(Q_7) \leq \gamma(Q_7) + \chi(Q_7) = 18$.

3. Graphs with $\chi_d(G) = \chi(G)$

For any graph G, we have $\chi_d(G) \geq \chi(G)$. In this section we investigate graphs for which $\chi_d(G) = \chi(G)$. In particular, we characterize unicyclic graphs, split graphs and complements of bipartite graphs with $\chi_d(G) = \chi(G)$.

It follows from Theorem 1.2 that for a tree T of order $n \geq 2$, $\chi_d(T) = \chi(T)$ if and only if $T = K_{1,n-1}$. In the following theorem we characterize unicyclic graphs with $\chi_d = \chi$.

Theorem 3.1. *Let G be a connected unicyclic graph. Then $\chi_d(G) = \chi(G)$ if and only if G is isomorphic to C_3 or C_4 or C_5 or the graph obtained from C_3 by attaching any number of leaves at one or two vertices of C_3.*

Proof. Let G be a unicyclic graph with $\chi_d(G) = \chi(G)$. Let C be the unique cycle in G. If C is an even cycle, then $\chi_d(G) = \chi(G) = 2$ and hence it follows Theorem 1.2 that $G = C_4$. Suppose C is an odd cycle, so that $\chi_d(G) = \chi(G) = 3$.

Suppose there exists a support vertex v not on C. Since there exists a χ_d-coloring of G in which $\{v\}$ is a color class, it follows that $\chi_d(G) \geq 4$, which is a contradiction. Hence any support vertex lies on C and any vertex not on C is a leaf. Since there exists a χ_d-coloring of G in which every support vertex appears as a singleton color class, it follows that the number of support vertices is at most two. Hence if C is of length three, then G is isomorphic to C_3 or the graph obtained from C_3 by attaching any number of leaves at one or two vertices of C_3. Suppose the length of C is at least 5. If there exists a support vertex v on C, then there exists a χ_d-coloring $\{\{v\}, V_1, V_2\}$ of G such that V_1 contains all the leaves of G. Now, there exists a vertex w such that $w \in V_2$, w lies on C and w is not adjacent to v. Clearly w does not dominate any color class of G, which is a contradiction. Thus G has no support vertices and hence $G = C$. Now, it follows from Theorem 1.5 that $G = C_5$.

The converse is obvious. □

Chellali and Maffray [5] have obtained a characterization of split graphs G with $\chi_d(G) = \gamma(G) + 1$. In the following theorem we prove that $\chi_d(G) = \omega(G)$ or $\omega(G) + 1$ for any split graph G. Arumugam *et al.* [2] have used this theorem to prove that the dominator coloring problem is NP-complete even for split graphs.

Theorem 3.2. *Let G be a split graph with split partition (K, I) and $|K| = \omega$. Then $\chi_d = \omega$ or $\omega + 1$. Furthermore $\chi_d = \omega$ if and only if there exists a dominating set D of G such that $D \subseteq K$ and every vertex v in I is nonadjacent to at least one vertex in $K - D$.*

Proof. The coloring of G given by $\mathcal{C} = \{\{v\} : v \in K\} \cup \{I\}$ is a dominator coloring of G and hence $\chi_d \leq \omega + 1$. Thus $\chi_d = \omega$ or $\omega + 1$. Now suppose $\chi_d = \omega$. Let $\mathcal{C} = \{V_1, V_2, \ldots, V_\omega\}$ be a dominator coloring of G. Hence $|V_i \cap K| = 1$. Let $D = \{x : \{x\} \in \mathcal{C}\}$ and let $v \in I$. Any color class dominated by v is of the form $\{x\}$, where $x \in D$. Hence it follows that D is a dominating set of G. Also, if $V_i \cap K = \{x\}$ where $V_i \in \mathcal{C}$ and $v \in V_i$, then $x \in K - D$ and v is nonadjacent to x.

Conversely, suppose there exists a dominating set D of G such that $D \subseteq K$ and every vertex v in I is nonadjacent to at least one vertex in $K - D$. Now we assign colors $1, 2, \ldots, \omega$ to the elements of K and for any vertex v in I we choose a vertex x in $K - D$ which is nonadjacent to v and assign the color of x to v. This gives a dominator coloring of G with ω colors. □

Theorem 3.3. *Let $G = (X \cup Y, E)$ be a bipartite graph and $|X| \leq |Y|$. Then $\chi_d(\bar{G}) = \omega(\bar{G})$ or $\omega(\bar{G}) + 1$. Further $\chi_d(\bar{G}) = \omega(\bar{G})$ if and only if one of the following is true.*

(1) $\omega(\bar{G}) > |Y|$.
(2) $|X| < |Y|$ *and* $N_{\bar{G}}(x) \cap Y \neq \emptyset$ *for all* $x \in X$.
(3) $|X| = |Y|$, $N_{\bar{G}}(x) \cap Y \neq \emptyset$ *for all* $x \in X$ *and* $N_{\bar{G}}(y) \cap X \neq \emptyset$ *for all* $y \in Y$.

Proof. Since $\bar{G}$ is a perfect graph, it follows that $\chi(\bar{G}) = \omega(\bar{G})$. Hence $\omega(\bar{G}) \leq \chi_d(\bar{G})$. Let $\mathcal{C}$ be a χ-coloring of $\bar{G}$ using $\omega(\bar{G})$ colors. Clearly $|V_i| \leq 2$ for all $V_i \in \mathcal{C}$, and $|V_i| = 2$ if and only if $|V_i \cap X| = |V_i \cap Y| = 1$.

Case i. $\omega(\bar{G}) > |Y| \geq |X|$.

Since $|X| \leq |Y| < \omega(\bar{G})$, there exist two color classes V_1 and V_2, such that $V_1 \cap X = \emptyset$ and $V_2 \cap Y = \emptyset$. Hence $|V_1| = |V_2| = 1$, so that $\mathcal{C}$ is a dominator coloring of $\bar{G}$. Thus $\chi_d(\bar{G}) = \omega(\bar{G})$.

Case ii. $\omega(\bar{G}) = |Y| > |X|$.

In this case there exists y_j in Y such that $\{y_j\} \in \mathcal{C}$ and hence every element of Y dominates the color class $\{y_j\}$. Now, suppose $N_{\bar{G}}(x) \cap Y \neq \emptyset$ for all $x \in X$. Let

$y_x \in N_{\bar{G}}(x) \cap Y$. Then x dominates the color class which contains y_x. Thus $\mathcal{C}$ is a dominator coloring of $\bar{G}$ and $\chi_d(\bar{G}) = \omega(\bar{G})$. Now suppose there exists $x \in X$ such that $N_{\bar{G}}(x) \cap Y = \emptyset$. Then in any dominator coloring of $\bar{G}$, the color class which x dominates is of the form $\{x_1\}$ where $x_1 \in X$ and hence it follows that $\chi_d(\bar{G}) \geq \omega + 1$. Now let $V_1 \in \mathcal{C}$ and $x \in V_1$. Clearly $|V_1| = 2$. Let $V_1 = \{x, y\}$. Then $\mathcal{C}_1 = (\mathcal{C} - \{v_1\}) \cup (\{x\}, \{y\})$ is a χ_d-coloring of $\bar{G}$ and hence $\chi_d(\bar{G}) \leq \omega(\bar{G}) + 1$. Thus $\chi_d(\bar{G}) = \omega(\bar{G}) + 1$.

Case iii. $\omega(\bar{G}) = |Y| = |X|$.

Then $|V_i| = 2$ for each $V_i \in \mathcal{C}$. Now if $N_{\bar{G}}(x) \cap Y \neq \emptyset$ for all $x \in X$ and $N_{\bar{G}}(y) \cap X \neq \emptyset$ for all $y \in Y$, then $\mathcal{C}$ itself is a dominator coloring of $\bar{G}$ so that $\chi_d(\bar{G}) = \omega(\bar{G})$. Otherwise proceeding as in Case ii, we get $\chi_d(\bar{G}) = \omega(\bar{G}) + 1$. □

4. Graphs with minimum degree 1

In this section we consider graphs with $\delta(G) = 1$. The following theorem gives a lower bound for $\chi_d(G)$ in terms of the number of support vertices.

Theorem 4.1. *If G is a graph with $\delta(G) = 1$ and k support vertices, then $\chi_d(G) \geq k + 1$, and $\chi_d(G) = k + 1$ if and only if the set of nonsupport vertices is an independent dominating set of G.*

Proof. Let S denote the set of support vertices of G. Let $v \in S$. Then in any χ_d-coloring of G either v or a leaf adjacent to v appears as a singleton color class and hence it follows that $\chi_d(G) \geq k + 1$. Now, if $V - S$ forms an independent dominating set of G, then $\{\{v\} : v \in S\} \cup \{V - S\}$ is a dominator coloring of G, so that $\chi_d(G) = k + 1$.

Conversely, suppose $\chi_d(G) = k + 1$. Let $\mathcal{C}$ be a χ_d-coloring of G such that $\{v\}$ is a color class for every $v \in S$. Hence it follows that $V - S$ is a color class in $\mathcal{C}$ and hence $V - S$ is an independent dominating set of G. □

COROLLARY 4.2

If G is any graph of order n, then $\chi_d(G \circ K_1) = n + 1$.

Theorem 4.3. *Let G be a graph with $\delta(G) = 1$. Let V_1 be the set of all support vertices of G and let $|V_1| = k$. Then $\chi_d(G) = k + 2$ if and only if the following conditions are satisfied*:

(i) *$\langle V - V_1 \rangle$ is a nontrivial bipartite graph, and*
(ii) *if V_1 is not a dominating set, then $\langle V - V_1 \rangle$ contains exactly one nontrivial component which is a complete bipartite graph with $V_2 = V - N[V_1]$ as one of the partite sets.*

Proof. Suppose $\chi_d(G) = k + 2$. Let $\mathcal{C} = \{\{v\} : v \in V_1\} \cup \{C_1, C_2\}$ be a χ_d-coloring of G such that C_1 contains all the pendant vertices of G. Clearly $\langle V - V_1 \rangle$ is a nontrivial bipartite graph with bipartition C_1, C_2. Now, suppose V_1 is not a dominating set. Then $V_2 = V - N[V_1] \neq \emptyset$ and every vertex of V_2 dominates the color class C_2. It follows that $V_2 \subseteq C_1$ or $V_2 \subseteq C_2$. Hence $\langle V - V_1 \rangle$ contains exactly one nontrivial component which is a complete bipartite graph with V_2 as one of the partite sets.

Conversely, suppose (i) and (ii) are satisfied. Clearly $\chi_d(G) \geq k + 2$. Now if V_1 is a dominating set, then $\{\{v\} : v \in V_1\} \cup \{C_1, C_2\}$, where C_1, C_2 is a bipartition of $\langle V - V_1 \rangle$

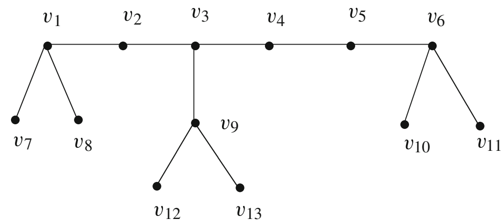

Figure 1. A tree with $\gamma = 4$ and $\chi_d = 5$.

is a χ_d-coloring of G. If V_1 is not a dominating set, then $\{\{v\} : v \in V_1\} \cup \{C_1, C_2\}$, where C_2 is the set consisting of V_2 and all the isolated vertices of $\langle V - V_1 \rangle$ is a χ_d-coloring of G. Hence $\chi_d(G) = k + 2$. □

Theorem 4.4. *Let G be any graph with $\delta(G) = 1$. Then $\chi_d(G) > \gamma(G)$.*

Proof. Let $\{V_1, V_2, \ldots, V_k\}$ be a χ_d-coloring of G in which every support vertex is a singleton color class and the set of all leaves of G is contained in one color class, say V_1. Let $S = \{v_2, v_3, \ldots, v_k\}$ where $v_i \in V_i, 2 \leq i \leq k$. Clearly S contains all the support vertices. We now claim that S is a dominating set of G. Let $v \in V - S$ and let v dominate the color class V_i. If $i > 1$, then v_i dominates v. If $i = 1$, then v is either a support vertex or a leaf and hence is dominated by S. Thus $\gamma(G) \leq |S| = \chi_d(G) - 1$. □

PROPOSITION 4.5

Let T be a tree of order n. If there exists a γ-set S in T such that $V - S$ is independent, then $\chi_d(T) = \gamma(T) + 1$.

Proof. It follows from Theorem 1.4 that $\chi_d(T) = \gamma(T) + 1$ or $\gamma(T) + 2$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a γ-set in T such that $V - S$ is independent. Then $\mathcal{C} = \{\{v_i\} : 1 \leq i \leq k\} \cup \{\{V - S\}\}$ is a χ_d-coloring of T and hence $\chi_d(T) = \gamma(T) + 1$. □

Remark 4.6. The converse of Proposition 4.5 is not true. For the tree T given in figure 1, the set $D = \{v_1, v_4, v_6, v_9\}$ is a minimum dominating set and $\mathcal{C} = \{\{v_1\}, \{v_6\}, \{v_9\}, \{v_3, v_5\}, \{v_2, v_4, v_7, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}\}$ is a χ_d-coloring of T. Hence $\gamma(T) = 4$, $\chi_d(T) = 5$. However, for any γ-set S in T, $V - S$ is not independent.

5. Dominator chromatic number of the Mycielskian

In search for triangle-free graphs with arbitrarily large chromatic number, Mycielski [19] gave an elegant graph transformation. For a graph $G = (V, E)$, the *Mycielskian* of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and is disjoint from V, and $E' = E \cup \{xy' : xy \in E\} \cup \{x'u : x' \in V'\}$. The vertices x and x' are called twins of each other and u is called the root of $\mu(G)$. For recent results on the Mycielskian of a graph, we refer to [3, 6–9, 17, 18]. The Mycielskian of C_5 along with a dominator coloring is given in figure 2.

It is well-known that $\chi(\mu(G)) = \chi(G) + 1$. However for dominator colorings, we have the following theorem.

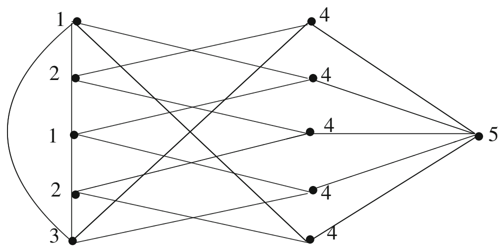

Figure 2. A dominator coloring of $\mu(C_5)$.

Theorem 5.1. *For any graph G*, $\chi_d(G) + 1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2$. *Further if there exists a* χ_d*-coloring* $\mathcal{C}$ *of G in which every vertex v dominates a color class* V_i *with* $v \notin V_i$, *then* $\chi_d(\mu(G)) = \chi_d(G) + 1$.

Proof. If $\mathcal{C}$ is any χ_d-coloring of G, then $\mathcal{C} \cup \{V', \{u\}\}$ is a dominator coloring of $\mu(G)$ and hence $\chi_d(\mu(G)) \leq \chi_d(G) + 2$. Now let $\chi_d(\mu(G)) = k$. Let $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ be a dominator coloring of $\mu(G)$ and let $u \in V_1$.

Case i. $V_1 = \{u\}$.

For each color α that appears on a vertex of V' but not on V, we choose an arbitrary vertex x' of V' that has the color α and recolor its twin $x \in V$ with color α. The restriction of this coloring to G gives a dominator coloring of G with $k - 1$ colors.

Case ii. $V_1 \neq \{u\}$.

Let $S = V_1 \cap V(G)$. Note that no vertex in $V(G)$ dominates the color class V_1. As in Case i, for each color α that appears on a vertex of V' but not on V, we choose an arbitrary vertex x' of V' that has the color α and recolor its twin $x \in V$ with color α. Now for each vertex $v \in S$, we recolor v with the color of its twin v'. Let $\mathcal{C}'$ denote the restriction of this coloring to $V(G)$. We first show that $\mathcal{C}'$ is a proper coloring of G. Let u and v be two adjacent vertices in G. If both u and v are in $V - S$, obviously they have distinct colors. If $u \in S$ and $v \in V - S$, then v is adjacent to u', and the new color of u is that of u'. Hence u and v have distinct colors.

Now, let $v \in V(G)$ and let V_i be the color class dominated by v in $\mathcal{C}$. Since $i \neq 1$, it follows that v continues to dominate the same color class in $\mathcal{C}'$. Thus $\chi_d(\mu(G)) > \chi_d(G)$, and hence $\chi_d(\mu(G)) = \chi_d(G) + 1$ or $\chi_d(G) + 2$.

Now, if there exists a χ_d-coloring of G in which every vertex v dominates a color class V_i with $v \notin V_i$, then the coloring of $\mu(G)$ obtained by assigning the color of v_i to its twin v_i' and a new color to the root u, is a dominator coloring of $\mu(G)$. Hence $\chi_d(\mu(G)) = \chi_d(G) + 1$. □

There exist graphs with $\chi_d(\mu(G)) = \chi_d(G) + 1$ or $\chi_d(\mu(G)) = \chi_d(G) + 2$.

For the complete graph K_n, we have $\chi_d(\mu(K_n)) = \chi_d(K_n) + 1 = n + 1$. Also for the cycle C_6, we have $\chi_d(\mu(C_6)) = \chi_d(C_6) + 1 = 5$.

The following lemma gives an example of a graph with $\chi_d(\mu(G)) = \chi_d(G) + 2$.

Lemma 5.2. $\chi_d(\mu(C_5)) = \chi_d(C_5) + 2$.

Proof. Let $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1)$. Since $\chi_d(C_5) = 3$, we need to prove that $\chi_d(\mu(C_5)) = 5$. Suppose $\chi_d(\mu(C_5)) = 4$. Let $\mathcal{C}$ be a dominator coloring of $\mu(C_5)$ using four colors 1, 2, 3 and 4. If all the colors appear on C_5, then the root u does not dominate a color class. Hence, we may assume that the vertices v_1, v_2, v_3, v_4 and v_5 receive respectively the colors 1, 2, 1, 2 and 3. If $\{v_5\} \in \mathcal{C}$, then v_5' receives color 4 and hence the root u receives color 1 or color 2. Hence color 4 appears on at least one of v_1' or v_4'. It follows that v_5' does not dominate a color class, a contradiction. Hence $\{v_5\} \notin \mathcal{C}$ and v_5 dominates the color class 4. Now v_5' receives color 3 and does not dominate any color class, a contradiction. Thus $\chi_d(\mu(C_5)) = 5$. □

6. Conclusion and scope

The following are some interesting problems for further investigation.

Problem 6.1. For which cycles C_n, $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$?

Problem 6.2. Characterize graphs G for which $\chi_d(G) = \chi(G)$ *or* $\chi_d(G) = \gamma(G)$.

Problem 6.3. Characterize trees T for which $\chi_d(T) = \gamma(T) + 1$.

Problem 6.4. Characterize graphs G for which $\chi_d(\mu(G)) = \chi_d(G) + 1$.

Problem 6.5. We have proved in [2] that the problem of determining the dominator chromatic number is NP-complete even for split graphs. Hence designing efficient algorithms for computing $\chi_d(G)$ for special families of graphs is an interesting problem. In particular does there exist a polynomial time algorithm for computing $\chi_d(T)$ for trees?

Problem 6.6. We observe that the dominator chromatic number of a graph G may increase arbitrarily on the removal of a vertex. For example $\chi_d(W_n) = \chi_d(C_{n-1} + K_1) = \chi(C_{n-1} + K_1) = 3$ or 4 according as n is odd or even and on removing the central vertex of W_n, χ_d increases arbitrarily. Hence the study of changing and unchanging of the dominator chromatic number on the removal of a vertex or an edge is an interesting problem for further investigation.

Acknowledgements

The first two authors acknowledge support from Ball State University. The first and third authors acknowledge support from the Department of Science and Technology, New Delhi, through the n-CARDMATH Project SR/S4/MS:427/07. They also thank Dr. Daniel Cranston for his helpful suggestions. The authors are also thankful to the anonymous referee for several helpful suggestions which led to substantial improvement in the paper.

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