STELLA MARIS COLLEGE (AUTONOMOUS) CHENNAI 600 086
(For candidates admitted from the academic year 2015-16 & thereafter)

SUBJECT CODE : 15MT/PC/LA24

M. Sc. DEGREE EXAMINATION, APRIL 2019
BRANCH I – MATHEMATICS
SECOND SEMESTER

COURSE : CORE
PAPER : LINEAR ALGEBRA
TIME : 3 HOURS **MAX. MARKS : 100**

Section-A
Answer ALL the questions **(5x2=10)**

1. If M is an R-module and A and B are its submodules, then prove that $A \cap B$ is a submodule of M.
2. Prove that the relation of similarity is an equivalence relation in $A(V)$.
3. Write the companion matrix of $f(x) = (5 + 3x - 2x^2 + 7x^3 + x^4) \in F[x]$.
4. Find the minimal polynomial of the linear operator T on R^2 which is represented in the standard basis by the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
5. Show that the matrix $A = (c)$ is orthogonal if and only if $c = \pm 1$.

Section-B
Answer any FIVE questions **(5x6=30)**

6. Let R be the ring of integers. Then show that the set of all even integers is a cyclic R-module. Also find its generators.
7. If V is n-dimensional vector space over F and if $T \in A(V)$ has all its characteristic roots in F, then prove that T satisfies a polynomial of degree n over F.
8. If the matrices A and B in F_n are similar in K_n, where K is an extension of F, then prove that A and B are already similar in F_n.
9. Let T be a linear operator on the finite-dimensional space V. Let $c_1, c_2, \dots, c_k$ be the distinct characteristic values of T and let W_i , be the space of characteristic vectors associated with the characteristic values c_i. If $W = W_1 + W_2 + \cdots + W_k$, then prove that $\dim W = \dim W_1 + \dim W_2 + \cdots + \dim W_k$.
10. If V is a complex vector space over and if f is a form on V such that $f(\alpha, \alpha)$ is real, then prove that f is Hermitian.
11. For any linear operator T on a finite-dimensional inner product space V prove that there exists a unique linear operator T^* on V such that $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all α, β in V.
12. Define nilpotent transformation. Also prove that if $T \in A(V)$ is nilpotent, then $\alpha_0 + \alpha_1 T + \cdots + \alpha_m T^m$, where $\alpha_i \in F$, is invertible if $\alpha_0 \neq 0$.

Section-C

Answer any THREE questions **(3x20=60)**

13. (a) Define direct sum on modules.
    (b) If R is a Euclidean ring, then prove that any finitely generated R-module M, is the direct sum of a finite number of cyclic submodules.
    (c) Prove that any finite abelian group is direct product(sum) of cyclic groups.

14. (a) If $T \in A(V)$ has all its characteristic roots in F, then prove that there is a basis of V in which the matrix of T is triangular.
    (b) Prove that two nilpotent linear transformations are similar if and only if they have same invariants.

15. (a) Let V and W be vector spaces over F and suppose that ψ is a vector space isomorphism of V onto W. Suppose that $S \in A_F(V)$ and $T \in A_F(W)$ are such that for any v in V, $(vS)\psi = (v\psi)T$. Then prove that S and T have same elementary divisors.
    (b) Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let T_1 and T_2 be the linear transformations induced by T on $V_1$and V_2, respectively. If the minimal polynomial of T_1 over F is $p_1(x)$ while that of T_2 is $p_2(x)$, then prove that the minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

16. (a) State and prove Caley-Hamilton theorem.
    (b) Find the minimal polynomial of $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ using Caley-Hamilton theorem.

17. (a) Let V be an inner product space and let $\beta_1, \beta_2, \dots, \beta_n$ be any independent vectors in V. Then prove that we can construct orthogonal vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ in V such that for each $k = 1,2,3 \dots, n$ the set $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a basis for the subspace spanned by $\beta_1, \beta_2, \dots, \beta_k$.
    (b) On a finite dimensional inner product of positive dimension prove that every self adjoint operator has a non-zero characteristic vector .